Codimension one foliation with pseudo-effective conormal bundle (PART 3)

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Recollections (from PART 1 and 2)

We are dealing with codimension one distribution $\mathcal{D}$ defined on a n-dimensional compact Kähler manifold X by $Ker(\Omega)$,

$$\Omega \in H^0(X, \Omega^1_X \otimes L^*)$$

where $L \in Pic(X)$ is pseudo-effective (psef). One can identify L with the conormal bundle $N^*\mathcal{D}$ of $\mathcal{D}$. By assumptions, there exists on $N^*\mathcal{D}$ a metric h, expressed in trivializing charts as

$$h(x, \Omega) = e^{-\varphi(x)}$$

where φ is psh and the (globally well defined) positive current $T_h = i\partial\bar{\partial}\varphi$ represents $c_1(N^*\mathcal{D})$.

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By Demailly's integrability Theorem, $\mathcal{D}$ is involutive, hence defines a foliation denoted by $\mathcal{F}$.
The integrability is easily derived from the following identity

$$\partial\varphi \wedge \Omega = -d\Omega \tag{1}$$

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3. The curvature current T_h is also $\mathcal{F}$ invariant ($T_h \wedge \Omega = 0$). Outside the singular locus $Sing(\mathcal{F})$ of $\mathcal{F}$, one can then locally write

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To summarize. one inherits two closed positive (1, 1) currents $\mathcal{F}$ invariant:

- The $(1,1)$ form with L^∞ coefficients η_h.
- The "curvature current" T_h of η_h, $[T_h] = c_1(N^*\mathcal{F})$

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Example:
Consider a linear foliation defined by a holomorphic one form on a two dimensional torus T . Let us perform a first blow-up π_0 at $p_0 \in T$ and let $E_0 = \pi^{-1}(p_0)$ be the exceptional divisor . The pull-back foliation $\pi_0^*\mathcal{F}$ is defined by the form $\omega_0 = {\pi_0}^*\omega$ vanishing exactly at a point $p_1 \in E_0$. The local model of the singularity at p_1 is given by $xdy + ydx$. After blowing-up p_1, one obtains a second exceptional divisor E_1. This is a -1 rational curve invariant by $\mathcal{F}$ and along which ${\pi_1}^*\omega_0$ vanishes with multiplicity 1. In other words, the conormal bundle of the corresponding foliation coincides with $\mathcal{O}(E_1)$.

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This is a situation where $c_1(N^*\mathcal{F})$ is psef but not nef and actually reduced to its *negative part* in its *Zariski decomposition*. Actually, the only closed positive which represents $c_1(N^*\mathcal{F})$ is $[D]$ (the integration current along D).

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We will assume from now on that $N^*\mathcal{F}$ is **nef** and that $c_1(N^*\mathcal{F}) \neq 0$.

Construction of a transverse hyperbolic metric

GOAL: We would like to exhibit a "distinguished" metric h on $N^*\mathcal{F}$ such that the curvature current T_h satisfies

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Reformulation: this amounts to choose the local psh weight φ of h such that (outside $Sing(\mathcal{F})$)

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The construction of h involves two steps:

First step: Equality 2 holds for every h at the cohomological level:

$$[T_h] = [\eta_h] \in H^{1,1}(X, \mathbb{R})$$

Actually, one observes that $T_h \wedge \eta_h = 0$ and consequently $[T_h].[\eta_h] = 0$. One concludes by Hodge index Theorem.

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Second step: Given T a closed positive current representing $c_1(N^*\mathcal{F})$, one can associate η_T unique such that $[\eta_T] = [T]$ and T is the curvature current of η_T. One can show that the map $T \mapsto \eta_T$ is continuous (in the weak topology). One then concludes by the fixed point Theorem of Leray-Schauder-Tychonoff.

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Remark

One can prove in addition (maximum principle) that such a metric is unique (up to multiplication by a constant factor).

Developping map and dynamics

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This means the following: the lift of $\mathcal{F}$ on the universal cover $\tilde{X}$ is defined as the fibers of a map:

$$f : \tilde{X} \mapsto \mathbb{D}$$

equivariant with respect to some representation $\rho : \pi_1(X) \mapsto Aut(\mathbb{D})$ called the *monodromy representation of* $\mathcal{F}$ (attached to the underlying transverse hyperbolic structure).

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In our setting, one can show that f is surjective with connected fibers, then implying that the dynamic of $\mathcal{F}$ is entirely determined by ρ.
Now, by using that the space of leaves is compact (non necessarily Hausdorff) combined with basic tools of Kähler geometry, one can prove that

1. Either $Im(\rho)$ is a cocompact lattice and consequently $\mathcal{F}$ is a fibration ($\Rightarrow \kappa(N^*\mathcal{F}) = 1$)
2. Either $Im(\rho)$ is dense and then $\kappa(N^*\mathcal{F}) = -\infty$ (use the Schwarzian derivative "trick")

It remains to analyse this last situation. We will also assume in the sequence that X is **projective**.

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To do this, note that the existence of f can be translated into the existence of a section

$$s : X \mapsto X_\rho$$

where X_ρ is the flat $\mathbb{P}^1$ bundle over X constructed from the representation ρ:

$$X_\rho = \tilde{X} \times \mathbb{P}^1 / (x, z) \to (\gamma.x, \rho(\gamma)(z))$$

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The horizontal fibration $\{dz = 0\}$ descends on X_ρ as a foliation $\mathcal{F}_{X_\rho}$ (Riccati foliation).
By constructon, $\mathcal{F}$ coincide with the restriction of $\mathcal{F}_{X_\rho}$ to X (identified with $s(X)$).

Rigidity of the monodromy representation and the Corlette-Simpson's Theorem

One recalls the context: $\mathcal{F}$ is a codimension one foliation on X projective with $N^*\mathcal{F}$ nef but $\kappa(N^*\mathcal{F}) = -\infty$. We have seen previously that this corresponds to the existence of a transversely hyperbolic structure whose monodromy representation $\rho : \pi_1(X) \mapsto Aut(\mathbb{D})$ has dense image.

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Lemma

Let $\mathcal{C}$ a curve and let

$$\rho' : \pi_1(\mathcal{C}) \mapsto Aut(\mathbb{D})$$

a representation. Then there is no morphism

$$\varphi : X \mapsto \mathcal{C}$$

such that ρ factors to ρ' through φ ($\rho' \circ \varphi_* = \rho$).

Proof.

Let $X_\rho \mapsto X$, $C_{\rho'} \mapsto C$ the flat $\mathbb{P}^1$ bundles respectively associated to the representation ρ and ρ' as constructed previously. We proceed by contradiction. The existence of the morphism φ can be translated into the existence of a morphism of $\mathbb{P}^1$ bundles $\Psi : X_\rho \mapsto C_{\rho'}$ such that

$$\mathcal{F}_{X_\rho} = \Psi^* \mathcal{F}_{\mathcal{C}_{\rho'}}.$$

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In particular, we obtain that ${\Psi_X}^* \mathcal{F}_{\mathcal{C}_{\rho'}} = \mathcal{F}$, where Ψ_X denotes the restriction of Ψ to X $(= s(X))$. The foliation $\mathcal{F}$ being minimal (the leaves are dense), this implies that Ψ_X is dominant, otherwise the image of Ψ would be reduced to a curve and then $\mathcal{F}$ would admit a rational first integral. On the other hand, none of the leaves of $\mathcal{F}_{\mathcal{C}_{\rho'}}$ is dense Indeed, the action of $Aut(\mathbb{D})$ on $\mathbb{P}^1$ fix a disk. This contradicts the minimality of $\mathcal{F}$. □

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From the description of the rigid case described in loc.cit , one deduce that there exists a morphism

$$\varphi : X \mapsto \mathbb{D}^n/\Gamma$$

such that $\rho = \rho_i \circ \varphi_*$ where ρ_i is the ith tautological representation of $\Gamma = \pi_1^{orb}(\mathbb{D}^n/\Gamma) \mapsto \Gamma_i$ defined by the action on the ith factor (corresponding to the monodromy representation of the foliation $\mathcal{F}_i$).

Comments (and conclusion)

- *In general (when $N^*\mathcal{F}$ is only psef), one may exclude an hypersurface H invariant by $\mathcal{F}$ in order to define the monodromy representation which then takes the form $\rho : \pi_1(X - H) \mapsto Aut(\mathbb{D})$.*

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- *There may exist other "tautological" representations of Γ of unitary type (hidden behind the arithmetic nature of Γ) but they are not relevant in our case, as the image of ρ is a dense subgroup of $Aut(\mathbb{D})$.*

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THANK YOU FOR YOUR ATTENTION!