Codimension one foliation with pseudo-effective conormal bundle (PART 2)

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Recollections (from PART 1)

We are dealing with codimension one distribution $\mathcal{D}$ defined on a n-dimensional compact Kähler manifold X by $Ker(\Omega)$,

$$\Omega \in H^0(X, \Omega_X^1 \otimes L^*)$$

where $L \in Pic(X)$ is pseudo-effective (psef).
By Demailly's integrability criterion, $\mathcal{D}$ is integrable.
One will set $\mathcal{F} := \mathcal{D}$.

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Example: $\Omega \in H^0(X, \Omega^1_X \otimes \mathcal{O}(-D))$, D effective. Ω can be viewed as as a holomorphic one form. Here, $L = D$ is psef and can be represented by $[D]$, the integration current of $(n-1, n-1)$ forms on D
In this case, integrability simply follows from d-closedness of Ω.

Foliations defined by holomorphic one forms

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We have natural maps

$$\Omega^1_X(X) \to H^1(X, \mathbb{C})$$

$$\begin{array}{ccc} H_1(X, \mathbb{Z})/tor & \to & \Omega^1_X(X)^* \\ \gamma & \to & \Omega \to \int_\gamma \Omega \end{array}$$

Set $\Gamma = Im(H_1(X, \mathbb{Z})/tor) \subset \Omega^1_X(X)^*$. It turns out that Γ is a cocompact lattice in $\Omega^1_X(X)^*$, so that the quotient

$$Alb(X) := \Omega^1_X(X)^*/\Gamma$$

is a compact torus and called the *Albanese torus* of X.

Fixed once for all a point x_0. Integration of forms on paths starting at x_0 and ending at $x \in X$ gives a morphism

$$alb_X : X \to Alb(X).$$

By construction, $alb_X^* : \Omega^1(Alb(X)) \to \Omega^1_X(X)$ is an isomorphism.
In particular, if $\mathcal{F} = Ker(\Omega)$, $\Omega \in \Omega^1_X(X)$, then $\mathcal{F} = alb_X^*(\mathcal{F}_{lin})$ where $\mathcal{F}_{lin}$ is a linear foliation on the torus, i.e given by translation of a codimension one subgroup.

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We would like now to enumerate some criterias which guarantee that $\mathcal{F}$ is defined by a fibration $X \to C$ over a curve C.

Theorem (Castelnuovo-De Franchis)

Let $\Omega_1, \Omega_2 \in \Omega^1_X(X)$ (Xcompact Kähler), $\mathbb{C}$ linearly independant and such that $\Omega_1 \wedge \Omega_2 = 0$. Consider the foliation $\mathcal{F}$ defined by $Ker(\Omega_1) = Ker(\Omega_2)$. Then $\mathcal{F}$ is tangent to a fibration $f : X \to C$, where C is a curve of genus $g(C) \geq 2$.

Proof.

By assumptions, there exists a non constant meromorphic function $f : X \dashrightarrow \mathbb{P}^1$ such that $\Omega_1 = f\Omega_2$. The d-closedness of Ω_i then implies that $\Omega_1 \wedge df = \Omega_2 \wedge df = 0$. Moreover, one can easily check that f has no indeterminacy point (i.e is a genuine holomorphic map) and up to taking Stein factorization, one can suppose that $f : X \to C$ has connected fibers. In particular, $\Omega_1, \Omega_2 \in f^*(\Omega^1_C(C))$ and consequently $g(C) \geq 2$. □

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Remark

- *Assumptions of the Theorem* $\Leftrightarrow h^0(N^*_{\mathcal{F}}) \geq 2$.
- *If* $\Omega_i \in \Omega^1_X(X)$, $i = 1, 2, 3$ *are three non trivial section of* $N^*\mathcal{F}$ *and* f_1, f_2 *are the meromorphic function defined by* $\Omega_i = f_i\Omega_{i+1}, i = 1, 2$ *then* $df_1 \wedge df_2 = 0$, *i.e the forms* $df_1 \wedge df_2$ *are meromorphically independant or equivalently* $trdeg_{\mathbb{C}}(\mathbb{C}(f_1, f_2)) \leq 1$.

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Actually, one has a more general statement involving *Kodaira dimension*.

Definition

Let $L \in Pic(X)$ (X compact Kähler). The kodaira dimension of L, denoted by $\kappa(L)$ is defined as follows:
Consider the graded ring

$$R(L) := \bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$$

and the homogeneous field of fractions

$$Q(L) = \{\frac{l_i}{l_j} | l_i, l_j \in H^0(X, L^{\otimes m}, m \geq 0\}.$$

One then sets $\kappa(L) = trdeg_{\mathbb{C}} Q(X, L)$ if there exists $m > 0$ such that $h^0(L^{\otimes m}) \neq 0$ and $\kappa(L) = -\infty$ otherwise. Note that $\kappa(L) \leq n := dim(X)$ and that equality holds if L is ample.

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Theorem (Bogomolov)

Let X be a compact Kähler manifold and a non trivial twisted form $\Omega \in H^0(X, \Omega^1_X \otimes L^*)$, then $\kappa(L) \leq 1$ and when equality holds, $Ker(\Omega)$ is involutive and tangent to a fibration $X \to C$.

Sketch of proof.

If $\kappa(N^*_{\mathcal{F}}) \geq 1$, one comes back to the situation depicted in Castelnuovo-De Franchis Theorem, up to taking some ramified cover, . $\kappa(N^*_{\mathcal{F}}) > 1$ does not occur by the second point of the previous remark. □

Abundance defect

Remark

$\kappa(L) \geq 0 \Leftrightarrow h^0(L^{\otimes m}) \neq 0$ *for some* $m \Longrightarrow L$ *is psef*

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Yes!

Consider $X = \mathbb{D}^n/\Gamma$ where $\Gamma < Aut^0(\mathbb{D}^n) = Aut(\mathbb{D})^n$ is a torsion free irreducible cocompact lattice . X is endowed with n codimension one "tautological" foliations $\mathcal{F}_i$ whose lift on $\mathbb{D}^n$ is defined as the kernel of dz_i, $i = 1, ..., n$. These foliations are minimal (i.e have dense leaves), due to the irreducibility of the lattice.

Take for instance $i = 1$. Remark that $\mathcal{F}_1$ is also characterized as the kernel of the semi-positive $(1, 1)$ form

$$\eta = i \frac{dz_1 \wedge d\overline{z_1}}{(1 - |z_1|^2)}$$

which is nothing than that the area form of the Poincaré metric on the disk $\mathbb{D}_{z_1}$.

Note that η is well defined on X and induces on $N^*_{\mathcal{F}_1}$ a metric whose curvature is η itself (up to some positive factor): this is the dual meaning of the constant negative curvature of the Poincaré metric.

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Reformulation: $\mathcal{F}_1$ is equipped with a transversely hyperbolic structure: for a sufficiently fine open cover (U_i) of X, $\mathcal{F}_i$ is defined on each U_i by the levels $\{f_i = c\}$, $f_i : U_i \to \mathbb{D}$ subject on overlaps to the glueing conditions $f_i = \alpha_{ij}(f_j)$, $\alpha_{ij} \in Aut(\mathbb{D})$.

Actually, $f_i = z_1 \circ {\pi_i}^{-1}$ where ${\pi_i}^{-1}$ is a local inverse of $\pi : \mathbb{D}^n \to X$.

Let us justify roughly why $\kappa(N^*_{\mathcal{F}_1}) = -\infty$, assuming for simplicity that $h^0(N^*_{\mathcal{F}_1}) \neq 0$. This means that $\mathcal{F}_1$ is also defined as the kernel of a holomorphic (hence closed) one form ξ. On U_i, one can write $\xi = dF_i$. One can then check, that the collection of Schwarzian derivative $\{f_i, F_i\}$ of f_i with respect to F_i glue together to produce a non constant meromorphic first integral of $\mathcal{F}_1$. This contradicts the density of leaves.

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Question

Let $\Omega \in H^0(X, \Omega^1_X \otimes L)$ whith $c_1(L) = 0$ (i.e, L is a flat line bundle). Does this implies that L is torsion ($\Longleftrightarrow \kappa(L) = 0$).

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Obviously, the answer is negative in general.e.g $X = C$, where C is a curve of genus ≥ 2.
However, one has the

Theorem (T, 2016)

Let Ω as above (X compact Kähler)

1. *Suppose that codim$\{\Omega = 0\} \geq 2$; then L is torsion*
2. *If L is not torsion, the foliation $\mathcal{F}$ defined by Ker(Ω) is given by a morphism $X \to C$ onto a curve C.*

Remark

The first item can be rephrased as $c_1(N_{\mathcal{F}}^*) = 0 \Longrightarrow N_{\mathcal{F}}^*$ *is torsion, where* $\mathcal{F} = Ker(\Omega)$ *(abundance phenomenon)*

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Question

Is it possible to give a "reasonable" description of codimension one foliation $\mathcal{F}$ *whose conormal bundle violates the abundance principle:*

$$N^*_{\mathcal{F}} \text{ psef and } \kappa(N^*_{\mathcal{F}}) = -\infty$$

Statement of the main Theorem

Theorem

Let $\mathcal{F}$ a codimension one foliation on a projective manifold X whose conormal bundle does not satisfy the abundance principle. Then there exists a morphism

$$\varphi : X \to \overline{\mathbb{D}^m/\Gamma}^{BB}$$

such that $\mathcal{F} = \varphi^* \mathcal{F}_i$ for some $i \in \{1, ..., m\}$.

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One explains the notations:

1. Γ is an irreducible lattice of $Aut(\mathbb{D})^m$
2. BB denotes the Baily-Borel compactification, which consists here in adding finitely many points (cusps).
3. $\mathcal{F}_i$ is the i^{th} tautological foliation (induced by dz_i).

Comments

- *There exists a logarithmic version of this Theorem (natural because it holds on smooth projective models of* $\overline{\mathbb{D}^m/\Gamma}^{BB}$ *)*
- *The projective variety* $\overline{\mathbb{D}^m/\Gamma}^{BB}$ *somehow plays the role of the Albanese torus when* $\kappa(N^*_{\mathcal{F}}) \geq 0$

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We indicate briefly the ingredients of the Proof (details will be given in the last lecture)

1. One construct a hyperbolic transverse invariant metric with respect to $\mathcal{F}$ well defined on $X - H$ where H is a certain hypersurface $\mathcal{F}$-invariant. This produces a morphism $\pi_1(X - H) \to Aut(\mathbb{D})$, namely the monodromy representation attached to this hyperbolic transverse structure.
2. To conclude, one exploits fundamental results by Corlette and Simpson about rank 2 representations of quasi-projective groups.