Codimension one foliation with pseudo-effective conormal bundle (PART 1)

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General context:holomorphic distributions

A holomorphic distribution $\mathcal{D}$ on a complex manifold X is determined by a coherent subsheaf $T\mathcal{D}$ of TX such that TX/TD is torsion free.
The locus where $TX/T\mathcal{D}$ is not locally free is the singular locus of $\mathcal{D}$ denoted here by $Sing(\mathcal{D})$.
This condition above is imposed to avoid the existence of removable singularities of codimension one, it is thus equivalent to $codim(Sing(\mathcal{D})) \geq 2$.

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Alternative definition: One can also regard a distribution as the kernel (or annihilator) subsheaf) of a twisted non trivial q-form $\Theta \in H^0(X, \Omega^q_X \otimes L)$, $L \in Pic(X)$., saying that

$$X \in T\mathcal{D} \iff i_X \Theta = 0.$$

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Theorem/Definition

$\mathcal{D}$ is a foliation (on $X - Sing(\mathcal{D})$) $\Longleftrightarrow$ $T\mathcal{D}$ is involutive: $[T\mathcal{D}, T\mathcal{D}] \subset T\mathcal{D}$

In this situation, one inherits a partition of $X - Sing(\mathcal{D})$ into leaves.

The codimension one case

We are interested in codimension one distribution.
It is determined by a $\Theta \in H^0(X, \Omega^1_X \otimes L^*)$ where one can suppose that Θ does not vanish in codimension one.
In other words, it corresponds to the datum of a saturated invertible subsheaf $N^*\mathcal{D}$ of $\Omega^1_X(X)$: the **conormal** sheaf of $\mathcal{D}$ and $N^*\mathcal{D} \simeq L$ as a line bundle.

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Concretely, $\mathcal{D}$ is defined, from the datum of a sufficiently open cover $(U)_{U\in\mathcal{U}}$ of X, by a collection of one forms $\Theta_U \in \Omega^1_X(U)$ which generate $N^*\mathcal{D}$ over U and which glue together on the intersections $U \cap V$ by a multiplicative cocycle g_{UV}, $(g_{UV}) \sim L^*$:

$$\omega_U = g_{UV}\omega_V$$

In case $\mathcal{D}$ is a foliation the involutiveness property tranlates as the *integrability* condition

$$\Theta \wedge d\Theta = 0$$

.

By the way, note in general that $\Theta \wedge d\Theta$ makes sense as an element of $H^0(X, \Omega_X^3 \otimes L^{*\otimes 2})$. Indeed, in a trivializing chart, one can write $d(g\Theta) = dg \wedge \theta + gd\theta$, so that $(g\Theta) \wedge d(g\Theta) = g^2 \Theta \wedge d\Theta$.

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Example: take $L = \mathcal{O}(D)$ where D is an integral effective divisor. Then Θ can be regarded as a holomorphic *global* one form admitting D as its divisor of zeroes. In particular, if one supposes in addition that X is Kähler compact, this implies that $\mathcal{D}$ is integrable (i.e is a foliation) as a straighforward consequence of the following classical result.

Theorem

Let X be a n dimensional compact Kähler manifold and $\Theta \in H^0(X, \Omega_X^p)$. Then $d\Theta = 0$ (Θ is closed).

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Proof.

$(p = 1)$ Let ω be a Kähler form (i.e ω is a closed positive $(1,1)$ form). Remark that the (n,n) form $\Omega := d(\Theta \wedge d\bar{\Theta} \wedge \omega^{n-2})$ is both exact and nonnegative and $\not\equiv 0$ whenever $d\omega \not\equiv 0$. On the other hand, Stokes Theorem implies that $\int_X \Omega = 0$, whence the conclusion. □

Remark

Let X a projective manifold equipped with a codimension one distribution $\mathcal{D}$, then

- $N_{\mathcal{D}}^*$ *is never effectif if X is rationally connected.*
- N_D^* *is anti-ample if $Pic(X) = \mathbb{Z}$.*

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Non integrable examples of codimension one distribution are provided by contact manifolds X:
$dim(X) = n = 2m + 1$ $(m \geq 1)$ equipped with a contact form
$\Theta \in H^0(\Omega^1_X, L^*)$ such that $\underbrace{\Theta \wedge (d\Theta)^m}_{\in H^0(X, KX \otimes L^{*\otimes m+1})} \neq 0$ everywhere.

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Standard examples are given by given projectivized hyperplane bundles $\mathbb{P}(T^*Y)$. The contact structure (equivalently the corresponding hyperplane distribution $\mathcal{D}$) is tautologically defined saying that $v \in T_{[x]}\mathbb{P}(T^*Y)$ is tangent to $\mathcal{D}$ if $\pi_*(v) \in Ker(x)$ where $\pi : \mathbb{P}(T^*Y) \to Y$ is the canonical projection. Indeed, have in mind that $x \in T^*_y$, where $y = \pi([x])$.
Here, one has $L = \mathcal{O}_{\mathbb{P}(T^*Y)}(-1)$.

Demailly's integrability criterion

Let $\mathcal{D}$ a codimension one distribution on X compact Kähler.
Recall $N^*\mathcal{D} = \mathcal{O}(D)$ (positivity condition) $\Longrightarrow$ $\mathcal{D}$ is integrable.
Does this integrability property holds under weaker positivity properties?
The following result due to Jean-Pierre Demailly answers by the affirmative:

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Theorem (Demailly, 2000)

Let X be a compact Kähler manifold. Assume that there exists a pseudo-effective line bundle L on X and a non trivial holomorphic section $\Theta \in H^0(X, \Omega^p_X \otimes L^*)$. Then the distribution defined by $Ker(\Theta)$ is involutive.

Comments

Recall that a holomorphic line bundle on a compact complex manifold is said to be pseudo-effective (psef for short) if $c_1(L)$ *contains a closed positive* $(1,1)$ *current* T, *or equivalently, if* L *possesses a (possibly singular metric) hermitian metric* h *such that the curvature current* $T = \Theta_h(L) = -i\partial\bar{\partial}\log h$ *is nonnegative. If one expresses the norm defined by* h *in a trivializing chart as* $\|\xi\|_h = |\xi|^2 e^{-\varphi(x)}$, *this means that* $T = i\partial\bar{\partial}\varphi$ *and consequently that the local weight* φ *is a plurisubharmonic function.*

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Demailly's proof provides further informations (existence of an invariant transverse measure).

Corollary (Demailly, loc.cit)

Let X a compact Kähler manifold of dimension $n = 2m + 1$ admitting a contact structure, then KX is not psef, in particular the Kodaira dimension $\kappa(X)$ is equal to $-\infty$.

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Let $\Theta \in H^0(X, \Omega_1^X \otimes L^*)$ a contact form, then

$$\underbrace{\Theta \wedge d\Theta^m}_{without\ \ zeroes} \in H^0(X, KX \otimes L^{*\otimes m+1}).$$

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Together with previous results by Kebekus, Peternell, Sommese and Wiśniewski, this leads to the following characterization of contact projective manifolds:

Corollary (Demailly, loc.cit)

If X is a contact projective manifold with $b_2 \geq 2$, then $X = \mathbb{P}(T^*Y)$

"Proof" of Demailly's Theorem (p=1)

Recall that $\Theta \in H^0(X, \Omega_X^1 \otimes L^*)$ where L is psef. In a local trivialization of L, one has

$$\|\xi\|_h^2 = |\xi|^2 e^{-\varphi(x)}$$

where φ is psh. The dual norm on L^* reads as $\|\xi^*\|_{h^*}^2 = |\xi|^2 e^{\varphi(x)}$.

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$$\partial_\varphi f = \partial f + \partial\varphi \wedge f$$

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One has also the usual sesquilinear pairing $\{.,.\}_{h^*} : H^0(X, \Omega_X^{p,q} \otimes L^*) \times H^0(X, \Omega_X^{r,s} \otimes L^*) \to H^0(X, \Omega_X^{p+s,q+r})$ whose local expression is given by

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Now using the compatibility of ∇ with $\{.,.\}_{h^*}$, that is
$d\{f,g\}_{h^*} = \{\nabla f, g\}_{h^*} + (-1)^{deg(f)}\{f, \nabla g\}_{h^*}$ combined with the $\bar{\partial}$ closedness of Θ, one gets the following identity:

$$d\{\Theta, \partial_{h^*}\Theta\}_{h^*} = \{\partial_{h^*}\Theta, \partial_{h^*}\Theta\}_{h^*} + \{\Theta, \bar{\partial}\partial_{h^*}\Theta\}_{h^*} \tag{1}$$

To conclude, it suffices to remark that locally $\bar{\partial}\partial_{\varphi}\Theta = \bar{\partial}\partial\varphi \wedge \Theta$ wich allows to identify the last term in the RHS of (1) and obtains by Stokes Theorem that

$$\int_X \{\partial_{h^*}\Theta, \partial_{h^*}\Theta\}_{h^*} \wedge \omega^{n-2} = -\int_X T \wedge (i\{\Theta, \Theta\}_{h^*}) \wedge \omega^{n-2}$$

where ω is a Kähler form.

The form that we integrate on the LHS is nonnegative whereas the RHS is nonpositive (recall that T is positive). This clearly implies that $\partial_{h^*}\Theta = 0$ and then locally $d\Theta = \partial\Theta = -\partial\varphi \wedge \Theta$, whence the sough integrability. □

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Remark

One gets something much stronger than the sole integrability, namely that Θ is closed with respect to the Chern connection ∇: $\nabla\Theta = 0$. Note that one recovers the d- closedness when L is trivial.

Some results involving Demailly's Theorem

Positive answer to Beauville's conjecture in the totally decomposable case

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Theorem (Brunella, Pereira, T, 2006)

Let M be a compact Kähler manifold whose tangent bundle splits as a sum of line bundles

$$TM = L_1 \oplus ..... \oplus L_n.$$

Then the universal covering is isomorphic to a product of curves:

$$\tilde{M} = (\mathbb{P}^1)^r \times \mathbb{C}^s \times \mathbb{D}^s$$

Generalization of Rosenlicht's vanishing Theorem

Theorem (Loray, Pereira,T, 2018)

Let X be a compact Kähler manifold with KX psef and v a non trivial section of $\bigwedge^r TX$. Then $v(x) \neq 0$ everywhere.

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When X is projective, a fundamental result due to Boucksom, Demailly, Păun and Peternell (2013) asserts that

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Then, in the projective setting, the Theorem above generalizes to antisymmetric contravariant tensor fields the classical vanishing property established by Rosenlicht:

Theorem (Rosenlicht)

Let $V \in H^0(X, TX) - \{0\}$ where X is a projective manifold, Assume that there exists $x \in X$ such that $V(x) = 0$. Then X is uniruled.