BILLIARD ORBITS AND GEODESICS IN NON-INTEGRABLE FLAT DYNAMICAL SYSTEMS (PART II)

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C

1-direction geodesics in flat surfaces (in dimension 2)

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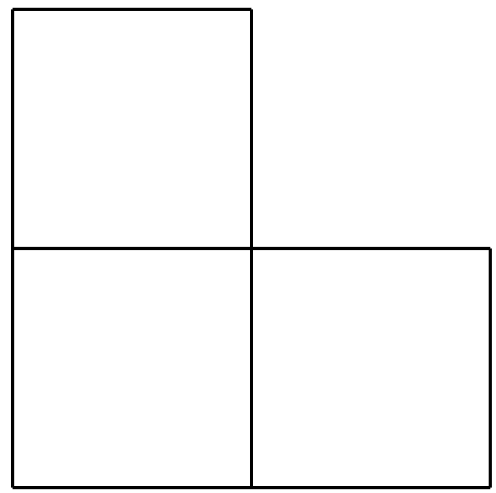
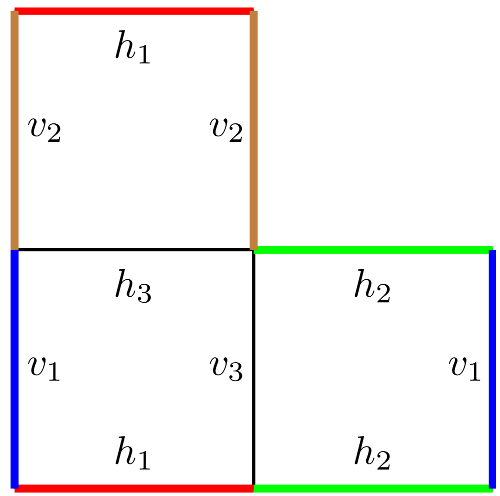
L-shape region



L-surface $\mathcal{P}$



1-direction geodesics in flat surfaces (in dimension 2)

geodesic on $\mathcal{P}$ of slope $\alpha = 1 + \sqrt{2} = [2; 2, 2, 2, \ldots] = 2 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$

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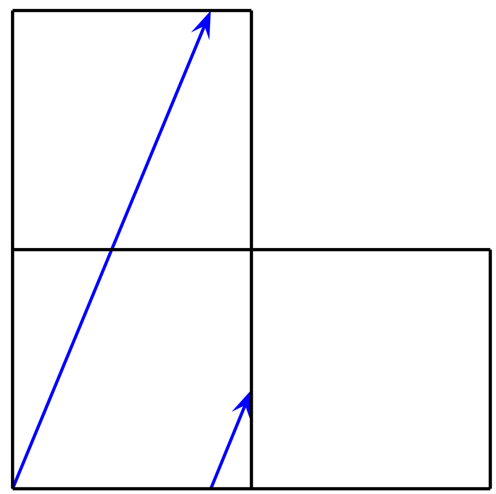
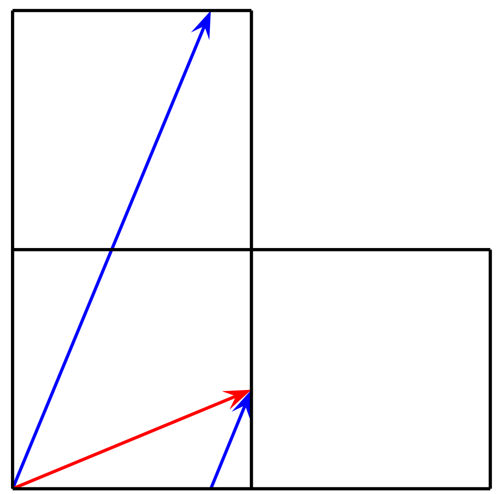


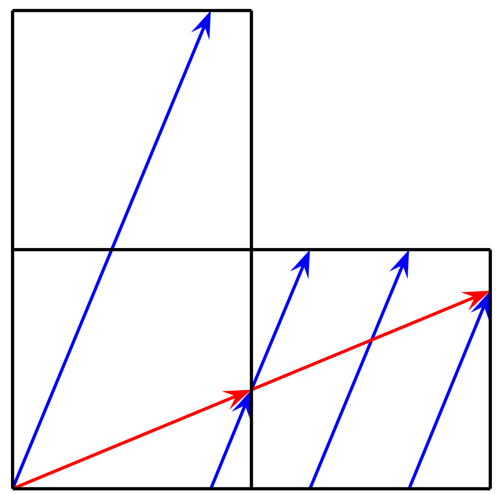
first detour crossing of a vertical street

geodesic on $\mathcal{P}$ of slope $\alpha = 1 + \sqrt{2} = [2; 2, 2, 2, \ldots] = 2 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$



first detour crossing of a vertical street and its shortcut

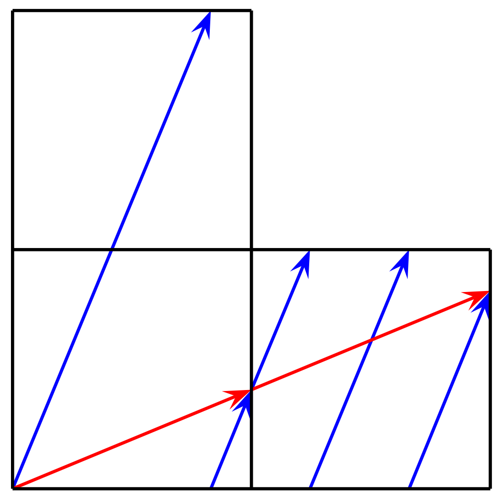
geodesic on $\mathcal{P}$ of slope $\alpha = 1 + \sqrt{2} = [2; 2, 2, 2, \ldots] = 2 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$



first detour crossing of a vertical street and its shortcut

second detour crossing of a vertical street and its shortcut

geodesic on $\mathcal{P}$ of slope $\alpha = 1 + \sqrt{2} = [2; 2, 2, 2, \ldots] = 2 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$



first detour crossing of a vertical street and its shortcut

second detour crossing of a vertical street and its shortcut

slope of shortcut is $\alpha - 2 = \sqrt{2} - 1 = \alpha^{-1} = [2, 2, 2, \ldots]$

almost vertical geodesic V of slope α

$\hookrightarrow$ almost horizontal shortline H of slope α^{-1}

assume V starts from some vertex of $\mathcal{P}$

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V and H are mutual shortlines

apply shortline process twice $\hookrightarrow$ back to original geodesic

assume V starts from some vertex of $\mathcal{P}$

almost vertical geodesic V of slope α

$\hookrightarrow$ almost horizontal shortline H of slope α^{-1}

almost horizontal geodesic H of slope α^{-1}

$\hookrightarrow$ almost vertical shortline V of slope α

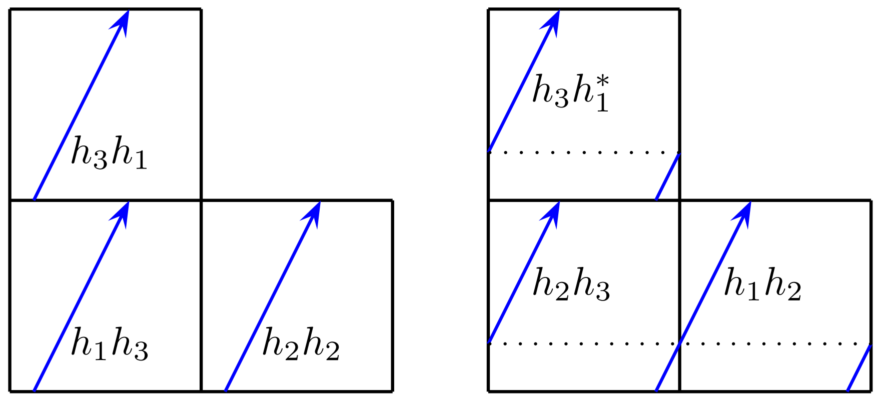
V and H are mutual shortlines

apply shortline process twice $\hookrightarrow$ back to original geodesic

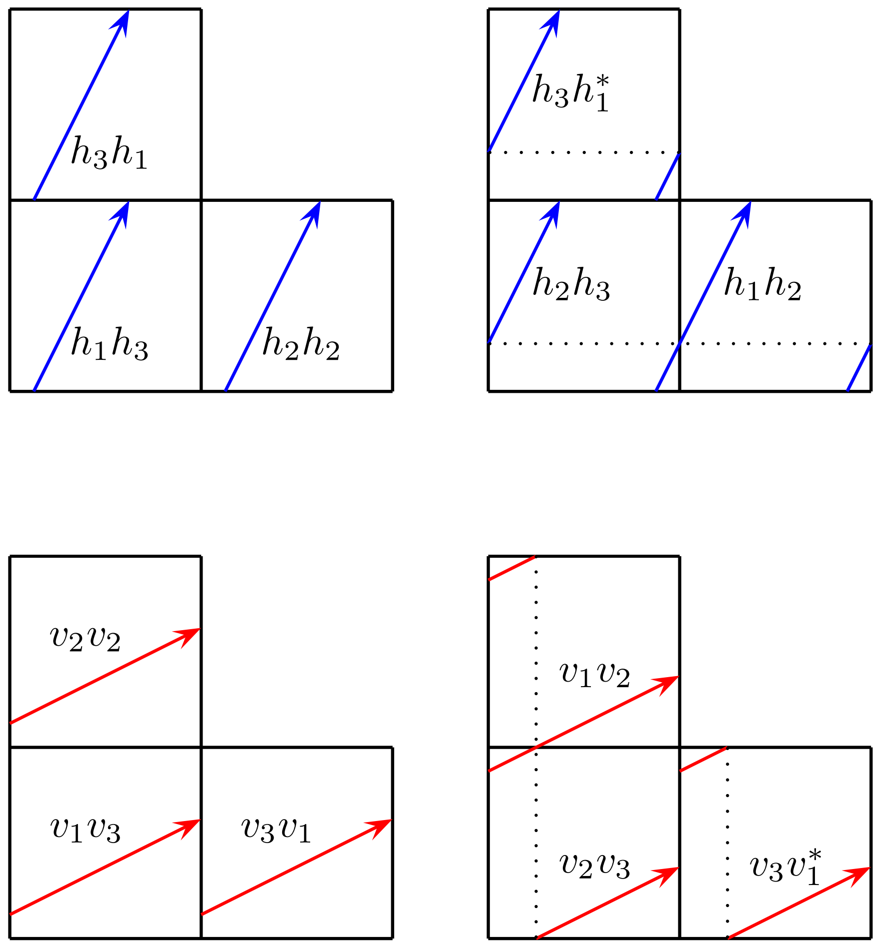
use 2-generation shortline to understand a geodesic

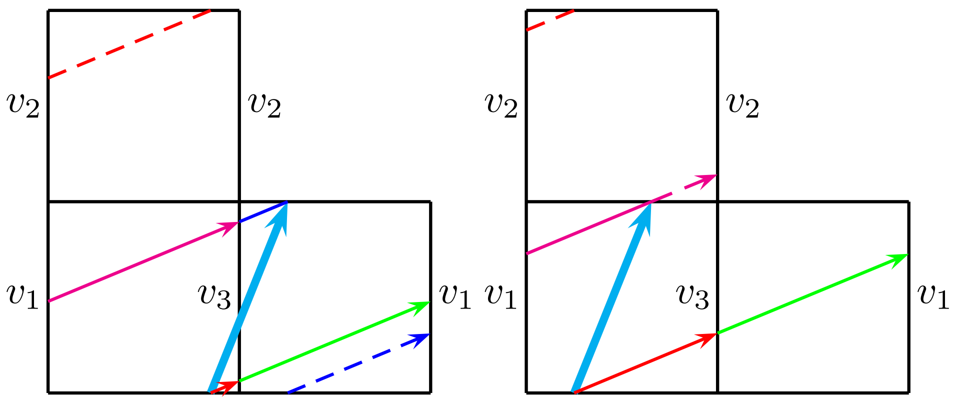
use 2-generation ancestor to understand a geodesic

almost vertical units of V

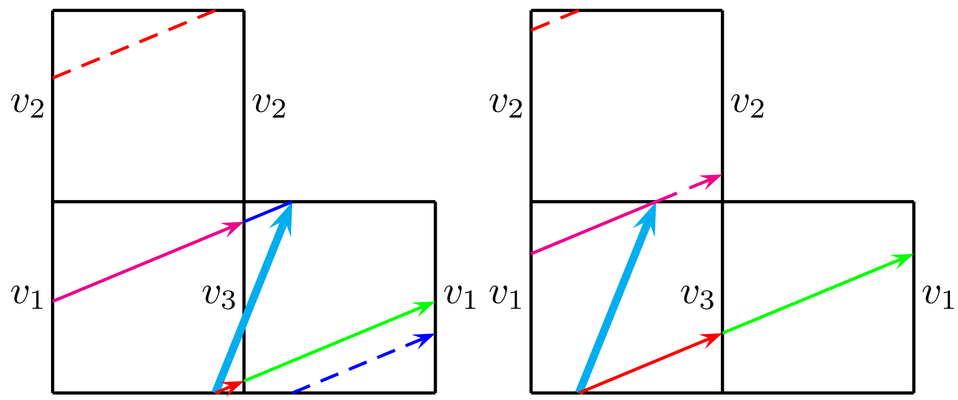


almost vertical units of V and almost horizontal units of H

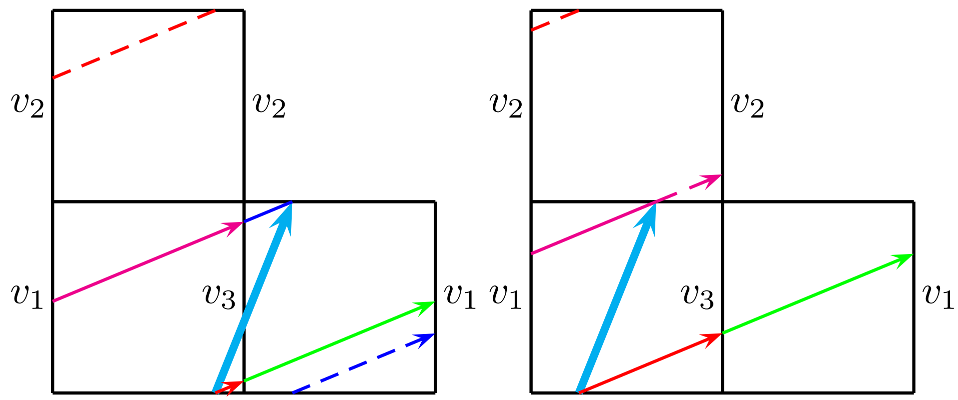




v_2
v_2
v_1
v_3
v_1
v_2
v_2
v_1
v_3
v_1



$h_1h_2 \hookrightarrow v_2v_3,\ v_3v_1,\ v_1v_3,\ v_3v_1^*$



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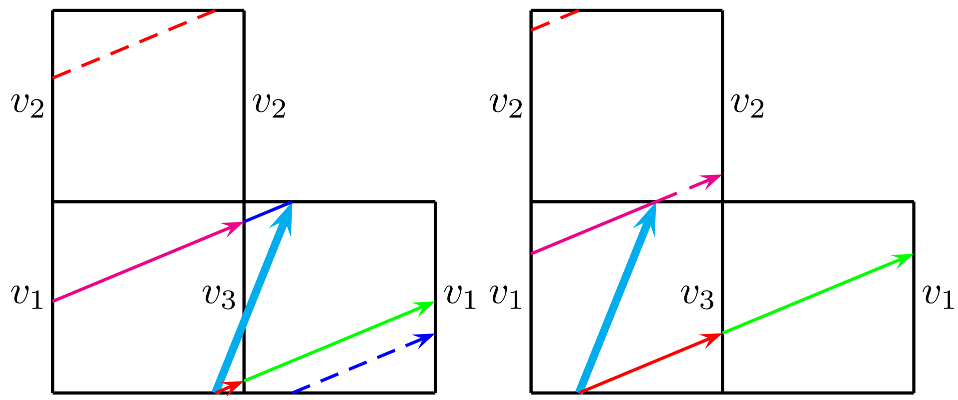
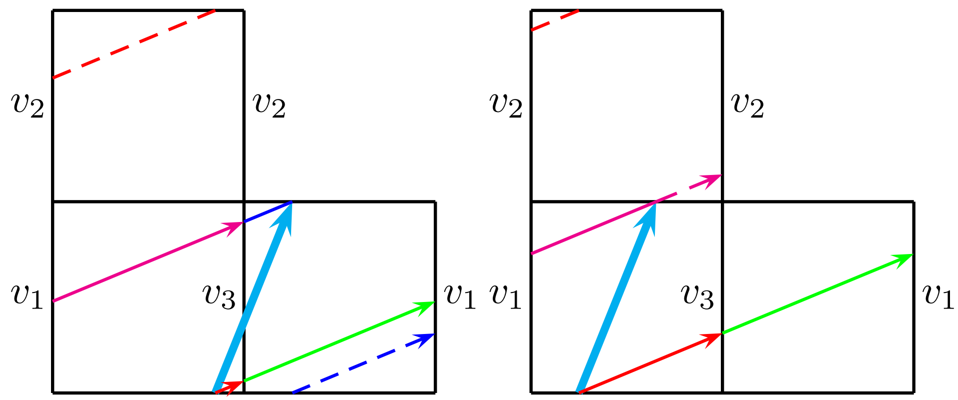


$$h_1h_2 \hookrightarrow v_2v_3,\ v_3v_1,\ v_1v_3,\ v_3v_1^* \qquad h_1h_3 \hookrightarrow v_2v_3,\ v_3v_1,\ v_1v_3$$

book keeping – Delete End Rule

$$h_1h_2 \rightharpoonup v_2v_3,\ v_3v_1,\ v_1v_3 \qquad h_1h_3 \rightharpoonup v_2v_3,\ v_3v_1$$



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book keeping – Delete End Rule

$h_1h_2 \rightharpoonup v_2v_3,\ v_3v_1,\ v_1v_3$ $\qquad$ $h_1h_3 \rightharpoonup v_2v_3,\ v_3v_1$

book keeping – Keep End Rule

$h_1h_2 \rightharpoonup v_3v_1,\ v_1v_3,\ v_3v_1^*$ $\qquad$ $h_1h_3 \rightharpoonup v_3v_1,\ v_1v_3$

$$h_1h_2 \rightharpoonup v_2v_3, v_3v_1, v_1v_3$$
$$h_1h_3 \rightharpoonup v_2v_3, v_3v_1$$
$$h_2h_2 \rightharpoonup v_3v_1^*, v_1v_3$$
$$h_2h_3 \rightharpoonup v_3v_1^*, v_1v_3, v_3v_1$$
$$h_3h_1 \rightharpoonup v_1v_2, v_2v_2$$
$$h_3h_1^* \rightharpoonup v_1v_2, v_2v_2, v_2v_2$$

$$h_1 h_2 \rightharpoonup v_2 v_3, v_3 v_1, v_1 v_3$$
$$h_1 h_3 \rightharpoonup v_2 v_3, v_3 v_1$$
$$h_2 h_2 \rightharpoonup v_3 v_1^*, v_1 v_3$$
$$h_2 h_3 \rightharpoonup v_3 v_1^*, v_1 v_3, v_3 v_1$$
$$h_3 h_1 \rightharpoonup v_1 v_2, v_2 v_2$$
$$h_3 h_1^* \rightharpoonup v_1 v_2, v_2 v_2, v_2 v_2$$

$$M_1 = \begin{array}{c} \\ h_1h_2 \\ h_1h_3 \\ h_2h_2 \\ h_2h_3 \\ h_3h_1 \\ h_3h_1^* \end{array} \begin{array}{c} \begin{array}{cccccc} v_1v_2 & v_1v_3 & v_2v_2 & v_2v_3 & v_3v_1 & v_3v_1^* \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Delete End Rule

$$h_1h_2 \rightharpoonup v_2v_3, v_3v_1, v_1v_3$$
$$h_1h_3 \rightharpoonup v_2v_3, v_3v_1$$
$$h_2h_2 \rightharpoonup v_3v_1^*, v_1v_3$$
$$h_2h_3 \rightharpoonup v_3v_1^*, v_1v_3, v_3v_1$$
$$h_3h_1 \rightharpoonup v_1v_2, v_2v_2$$
$$h_3h_1^* \rightharpoonup v_1v_2, v_2v_2, v_2v_2$$

Keep End Rule

$$v_1v_2 \rightharpoonup h_3h_1, h_1h_3, h_3h_1^*$$
$$v_1v_3 \rightharpoonup h_3h_1, h_1h_2$$
$$v_2v_2 \rightharpoonup h_1h_3, h_3h_1^*$$
$$v_2v_3 \rightharpoonup h_1h_3, h_3h_1, h_1h_2$$
$$v_3v_1 \rightharpoonup h_2h_2, h_2h_3$$
$$v_3v_1^* \rightharpoonup h_2h_2, h_2h_2, h_2h_3$$

Delete End Rule

$$M_1 = \begin{array}{c} \\ h_1h_2 \\ h_1h_3 \\ h_2h_2 \\ h_2h_3 \\ h_3h_1 \\ h_3h_1^* \end{array} \begin{array}{c} \begin{array}{cccccc} v_1v_2 & v_1v_3 & v_2v_2 & v_2v_3 & v_3v_1 & v_3v_1^* \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Keep End Rule

$$M_2 = \begin{array}{c} \\ v_1v_2 \\ v_1v_3 \\ v_2v_2 \\ v_2v_3 \\ v_3v_1 \\ v_3v_1^* \end{array} \begin{array}{c} \begin{array}{cccccc} h_1h_2 & h_1h_3 & h_2h_2 & h_2h_3 & h_3h_1 & h_3h_1^* \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$\mathbf{w}_0 = (0, 1, 0, 0, 0, 0)$ – first almost vertical unit is $h_1 h_3$

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numbers of $2k$-generation ancestor units described by $\mathbf{w}_k$

$\mathbf{w}_k = \mathbf{w}_0 (M_1 M_2)^k$

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numbers of $2k$-generation ancestor units described by $\mathbf{w}_k$

$\mathbf{w}_k = \mathbf{w}_0 (M_1 M_2)^k$

$\mathbf{w}_k^T = (M_2^T M_1^T)^k \mathbf{w}_0^T$

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2-step transition matrix $\mathcal{A} = M_2^T M_1^T = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 3 \\ 1 & 1 & 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix}$

$\mathbf{w}_0 = (0,1,0,0,0,0)$ – first almost vertical unit is $h_1 h_3$

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$\lambda_1 = (1+\sqrt{2})^2$, $\lambda_2 = \left(\frac{1+\sqrt{5}}{2}\right)^2$, $\lambda_3 = \ldots$, $\lambda_4 = \ldots$, $\lambda_5 = \ldots$, $\lambda_6 = \ldots$

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numbers of $2k$-generation ancestor units described by $\mathbf{w}_k$

$$\mathbf{w}_k = \mathbf{w}_0(M_1M_2)^k$$

$$\mathbf{w}_k^T = (M_2^TM_1^T)^k\mathbf{w}_0^T$$

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$\sqrt{\lambda_1} = [2;2,2,2,\ldots]$, $\sqrt{\lambda_2} = [1;1,1,1,\ldots]$

Beck–Donders–Yang (2020)

$L(t)$ – geodesic with slope $\alpha = 1 + \sqrt{2} = [2; 2, 2, 2, \ldots]$

$\mathcal{S}$ – arbitrary convex set on a square face of L-surface $\mathcal{P}$

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$$T \geqslant 2 \Rightarrow \left| \text{meas}\{t \in [0, T] : L(t) \in \mathcal{S}\} - \frac{\text{area}(\mathcal{S})T}{3} \right| = O\left(T^{\kappa_0}\right)$$

$$\kappa_0 = \frac{\log \frac{1+\sqrt{5}}{2}}{\log(1 + \sqrt{2})} = \frac{\log |\lambda_2|}{\log |\lambda_1|}$$

$\frac{1 + \sqrt{5}}{2} = [1; 1, 1, 1, \ldots]$ obtained from α by digit-halving

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$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, \ldots]$ obtained from α by digit-halving

error term is sharp

Beck–Donders–Yang (2020)

$L_k(t)$ – geodesic with slope $\alpha_k = k + \sqrt{k^2+1} = [2k; 2k, 2k, 2k, \ldots]$

$\mathcal{S}$ – arbitrary convex set on a square face of L-surface $\mathcal{P}$

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$$\kappa_0(k) = \frac{\log \frac{k+\sqrt{k^2+4}}{2}}{\log(k+\sqrt{k^2+1})}$$

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$$\kappa_0(k) = \frac{\log \frac{k+\sqrt{k^2+4}}{2}}{\log(k+\sqrt{k^2+1})} = \frac{\log|\lambda_2(k)|}{\log|\lambda_1(k)|}$$

$\lambda_1(k)$ and $\lambda_2(k)$ eigenvalues of $\mathcal{A}(k)$ with largest absolute values

Beck–Donders–Yang (2020)

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$\dfrac{k+\sqrt{k^2+4}}{2} = [k; k, k, k, \ldots]$ obtained from α_k by digit-halving

error term is sharp

Beck–Donders–Yang (2020)

$L_\gamma(t)$ – geodesic with slope $\gamma > 0$ quadratic irrational of the form

$\gamma = [2c_0; 2c_1, \ldots, 2c_h, 2a_1, \ldots, 2a_m, 2a_1, \ldots, 2a_m, \ldots]$

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$\mathcal{A}(\gamma)$ – some appropriate transition matrix

$\lambda_1(\gamma)$ and $\lambda_2(\gamma)$ are eigenvalues of $\mathcal{A}(\gamma)$ with largest absolute values

Beck–Donders–Yang (2020)

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$\lambda_1(\gamma)$ and $\lambda_2(\gamma)$ are eigenvalues of $\mathcal{A}(\gamma)$ with largest absolute values

irregularity exponent – $\kappa_0(\gamma) = \dfrac{\log|\lambda_2(\gamma)|}{\log|\lambda_1(\gamma)|}$

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$\lambda_1(\gamma)$ eigenvalue with larger absolute value of $\begin{pmatrix} 2a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 2a_m & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda_2(\gamma)$ eigenvalue with larger absolute value of $\begin{pmatrix} -a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -a_m & 1 \\ 1 & 0 \end{pmatrix}$

$\mathcal{S}$ – arbitrary convex set on a square face of L-surface $\mathcal{P}$

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$\circ\ \kappa_0(\gamma) > 0$

$$T \geqslant 2 \Rightarrow \left| \text{meas}\{t \in [0,T] : L_\gamma(t) \in \mathcal{S}\} - \frac{\text{area}(\mathcal{S})T}{3} \right| = O\left(T^{\kappa_0(\gamma)}\right)$$

error term is sharp

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error term is sharp

∘ $\kappa_0(\gamma) = 0$

$$T \geqslant 2 \Rightarrow \left| \text{meas}\{t \in [0,T] : L_\gamma(t) \in \mathcal{S}\} - \frac{\text{area}(\mathcal{S})T}{3} \right| = O\left((\log T)^2\right)$$

Beck–Donders–Yang (2020)

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Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – finite polysquare surface with street-LCM h

γ – quadratic irrational with all continued fraction digits divisible by h

$\mathcal{L}$ – 1-direction geodesic in $\mathcal{P}$ with slope γ

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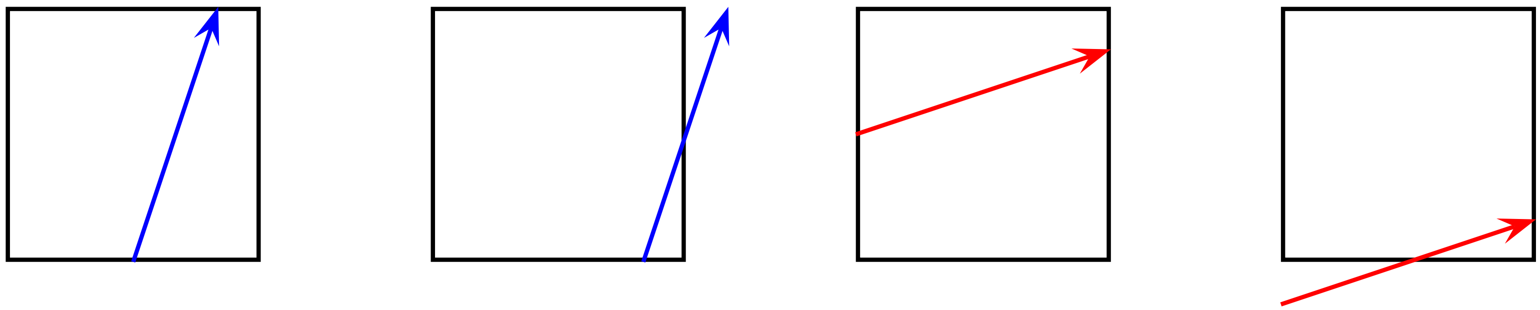
$\mathcal{A}(\gamma)$ – some appropriate transition matrix

$\lambda_1(\gamma)$ and $\lambda_2(\gamma)$ are eigenvalues of $\mathcal{A}(\gamma)$ with largest absolute values

irregularity exponent – $\kappa_0(\gamma) = \dfrac{\log|\lambda_2(\gamma)|}{\log|\lambda_1(\gamma)|}$

time-quantitative equidistribution of $\mathcal{L}$ with respect to all convex sets

2 types of almost vertical units and 2 types of almost horizontal units



square face of polysquare surface $\mathcal{P}$

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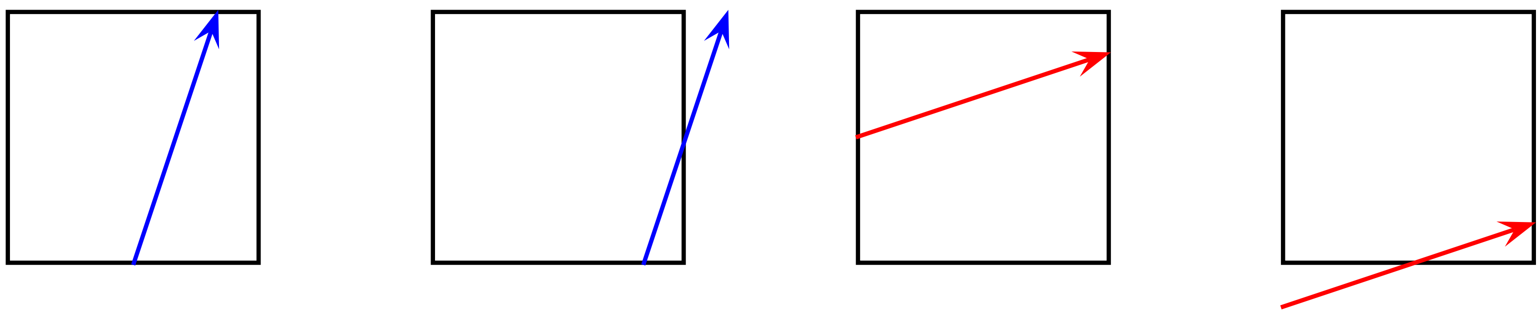

d − number of square faces of polysquare surface $\mathcal{P}$

$2d \times 2d$ transition matrix

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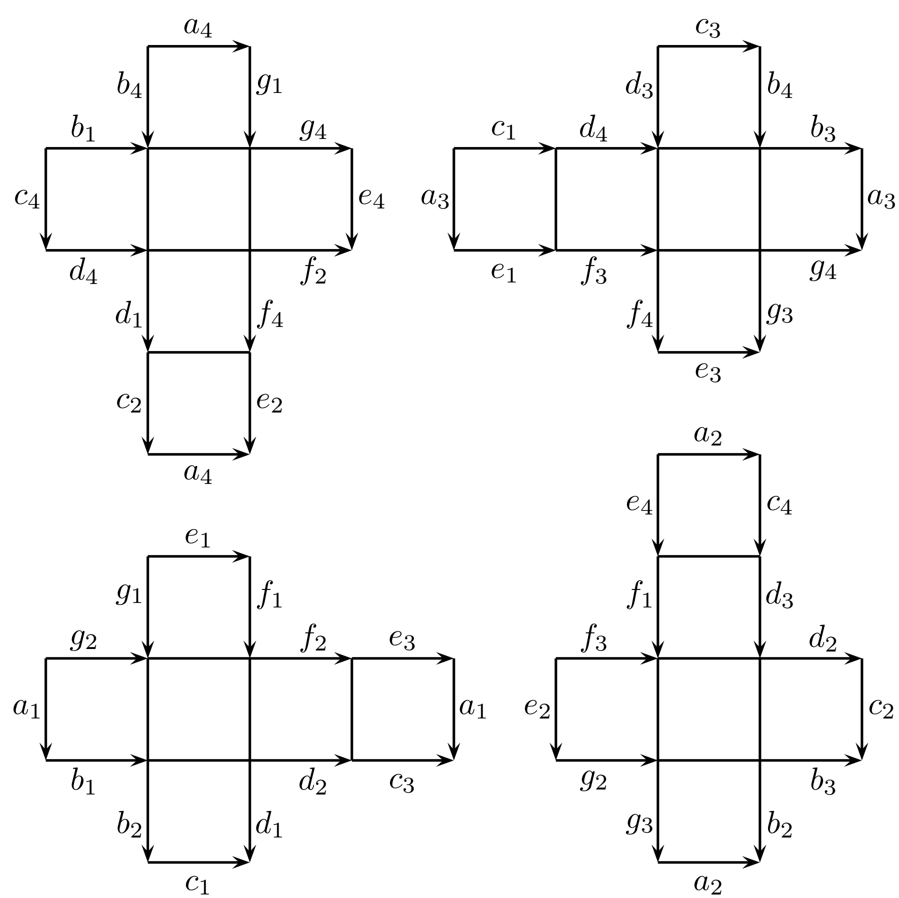
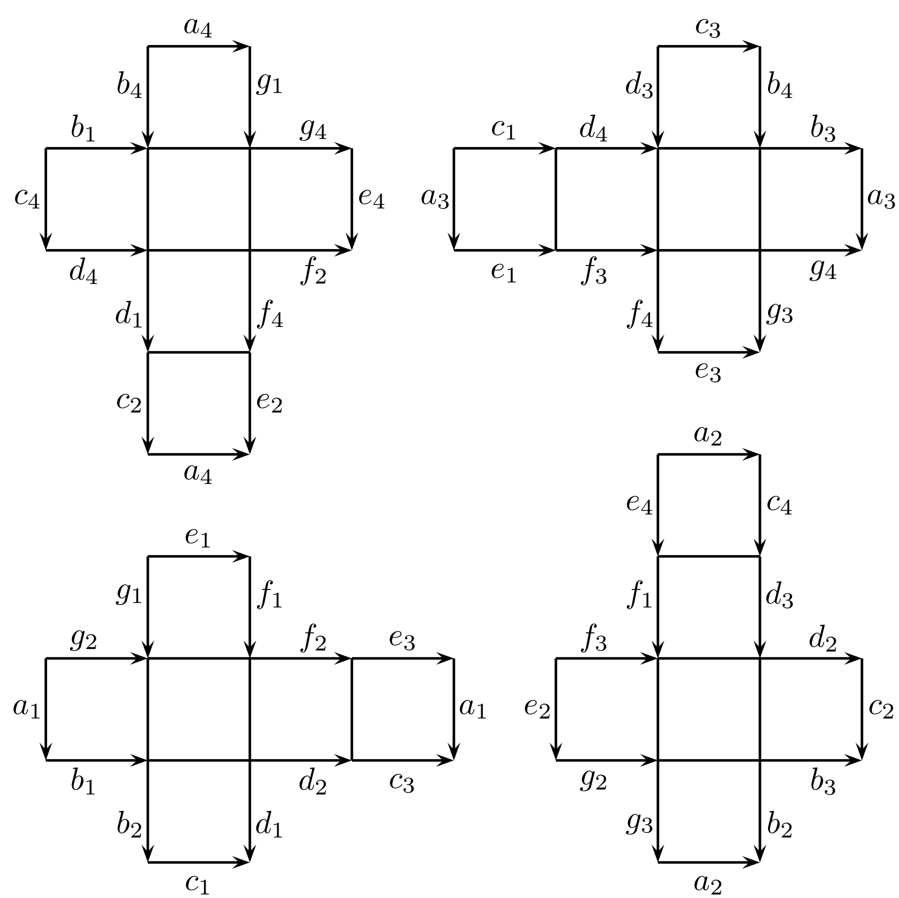
geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on surface



d – number of square faces of polysquare surface $\mathcal{P}$

$2d \times 2d$ transition matrix with $d = 24 \hookrightarrow 48 \times 48$ transition matrix

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on surface



polysquare surface $\mathcal{P}$ with d square faces

almost vertical geodesic V_0 from vertex in $\mathcal{P}$ of slope $\alpha = [n; m, n, m, \ldots]$

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V_0 made up of $2d$ types of almost vertical units of slope α

vector space W with basis $\mathscr{W} = \{2d$ almost vertical units$\}$

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back to W with basis $\mathscr{W}$

first step of ancestor process $\mathscr{W} \to \mathscr{W}'$

coefficient vectors taken as column vectors

$\hookrightarrow$ $2d \times 2d$ transition matrix M_1^T

first step of ancestor process $\mathscr{W} \to \mathscr{W}'$

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$\hookrightarrow$ $2d \times 2d$ transition matrix M_1^T

second step of ancestor process $\mathscr{W}' \to \mathscr{W}$

coefficient vectors taken as column vectors

$\hookrightarrow$ $2d \times 2d$ transition matrix M_2^T

first step of ancestor process $\mathscr{W} \to \mathscr{W}'$

coefficient vectors taken as column vectors

$\hookrightarrow$ $2d \times 2d$ transition matrix M_1^T

second step of ancestor process $\mathscr{W}' \to \mathscr{W}$

coefficient vectors taken as column vectors

$\hookrightarrow$ $2d \times 2d$ transition matrix M_2^T

2-step ancestor process $\mathscr{W} \to \mathscr{W}' \to \mathscr{W}$

$\hookrightarrow$ $2d \times 2d$ 2-step transition matrix $\mathcal{A} = M_2^T M_1^T$

$\mathcal{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_s$ with multiplicities $d_1, \ldots, d_s$

$|\lambda_1| \geqslant \ldots \geqslant |\lambda_s| \qquad d_1 + \ldots + d_s = 2d$

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$\mathbb{C}^{2d} = W_1 \oplus \ldots \oplus W_s$

W_i − $\mathcal{A}$-invariant subspace of $\mathbb{C}^{2d}$

containing eigenvector Ψ_i corresponding to eigenvalue λ_i

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basis $\Psi_{i,j}$, $i = 1, \ldots, s$, $j = 1, \ldots, d_i$, of $\mathbb{C}^{2d}$

V_0 starts at vertex of $\mathcal{P}$

with a finite succession of almost vertical units

with column coefficient vector $\mathbf{w}_0$ with respect to $\mathscr{W}$

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with column coefficient vector $\mathbf{w}_0$ with respect to $\mathscr{W}$

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assumption – $|\lambda_i| > 1 \Rightarrow$ basis of W_i consisting only of eigenvectors

$$|\lambda_i| \leqslant 1, \; i = s_0 + 1, \ldots, s$$

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main term and main error term – two largest eigenvalues

algorithm for finding crucial eigenvalues of $\mathcal{A}$

h − number of horizontal streets in polysquare surface $\mathcal{P}$

algorithm for finding crucial eigenvalues of $\mathcal{A}$

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can find $\mathcal{A}$-invariant subspace $\mathcal{V}$ of $\mathbb{C}^{2d}$

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– with $2h$ generators and explicitly given

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$h \times h$ street-spreading matrix $\mathbf{S}$

eigenvalues and eigenvectors of $\mathbf{S}$

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$h \times h$ street-spreading matrix $\mathbf{S}$ $\qquad \mathcal{A}|_{\mathcal{V}} = \begin{pmatrix} \mathbf{S} + I & I \\ \mathbf{S} & I \end{pmatrix}$

eigenvalues and eigenvectors of $\mathbf{S}$

$\hookrightarrow$ eigenvalues and eigenvectors of $\mathcal{A}|_{\mathcal{V}}$

algorithm for finding crucial eigenvalues of $\mathcal{A}$

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$\hookrightarrow$ relevant eigenvalues and eigenvectors of $\mathcal{A}$

billiard in right triangle with angle $\pi/8$

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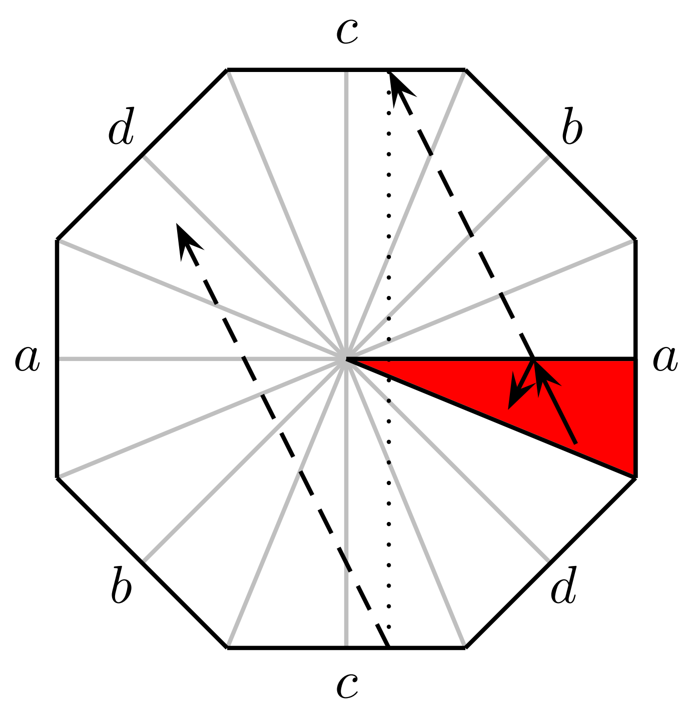

$\hookrightarrow$ 1-direction geodesic on regular octagon surface

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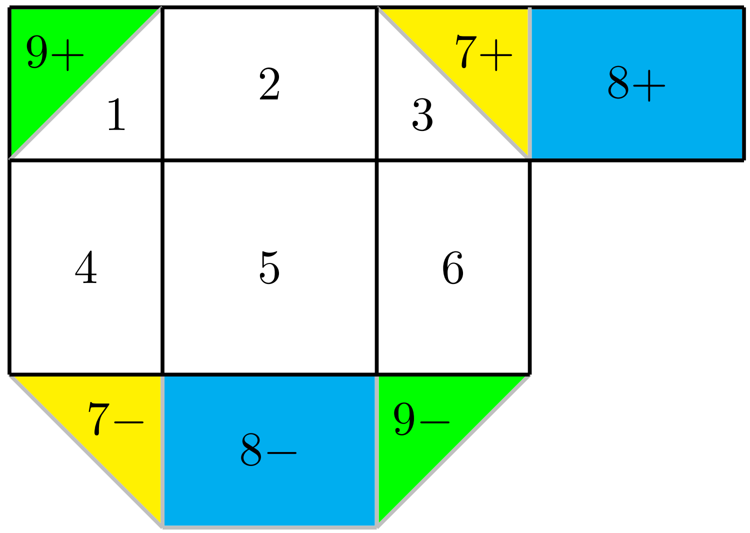

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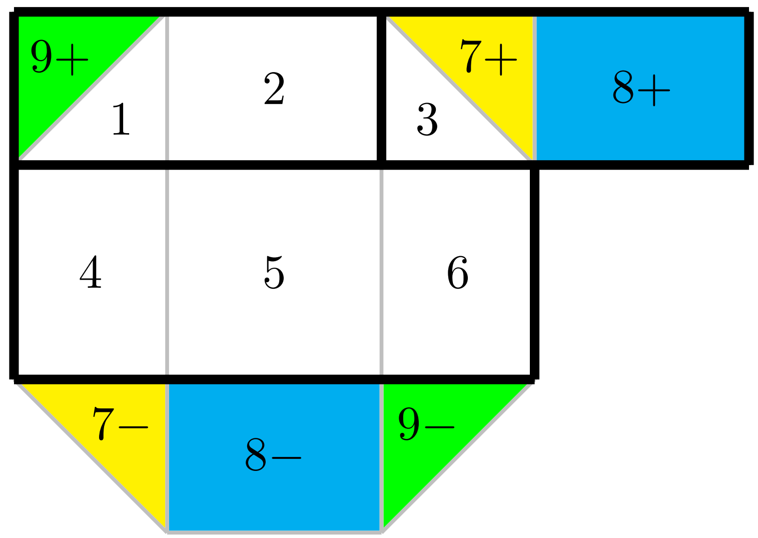

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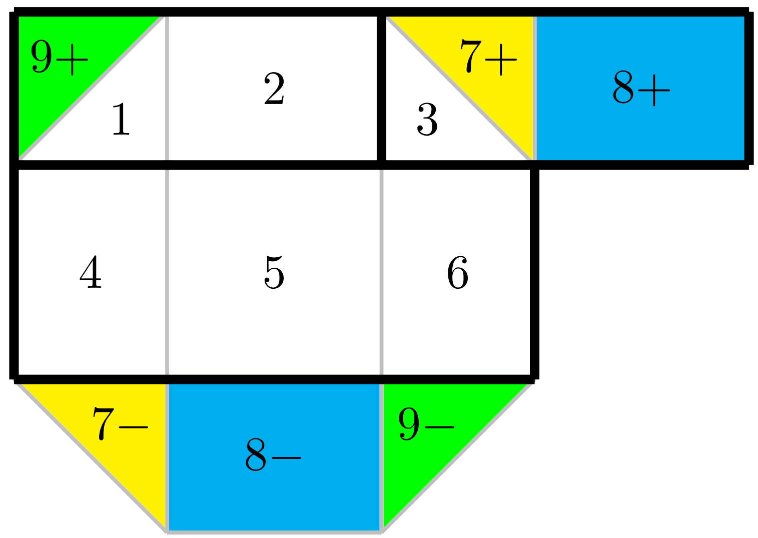
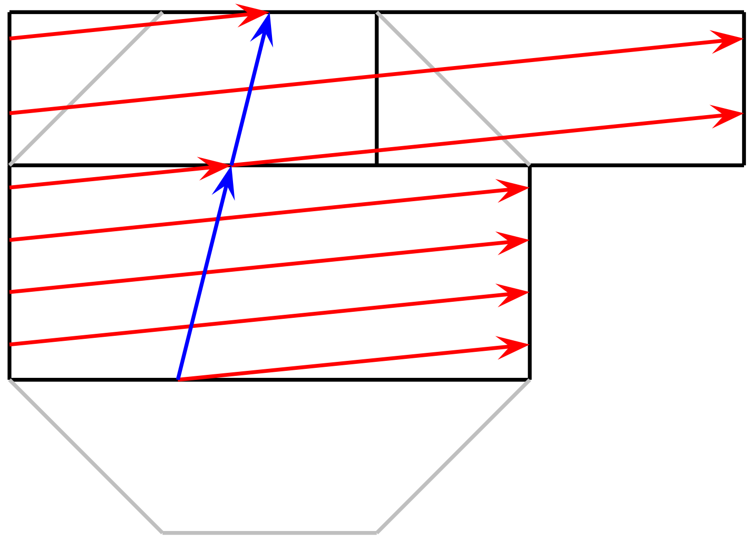


the 3 rectangles are similar

1-direction geodesic on regular octagon surface



the 3 rectangles are similar

almost horizontal detour crossings and almost vertical shortcuts

street-rational polyrectangle surface

$$\text{normalized horizontal street length} = \frac{\text{length of horizontal street}}{\text{width of horizontal street}}$$

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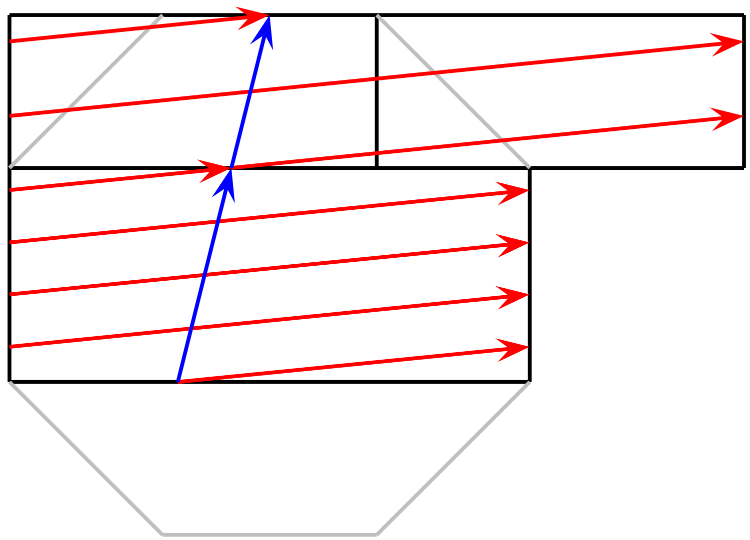
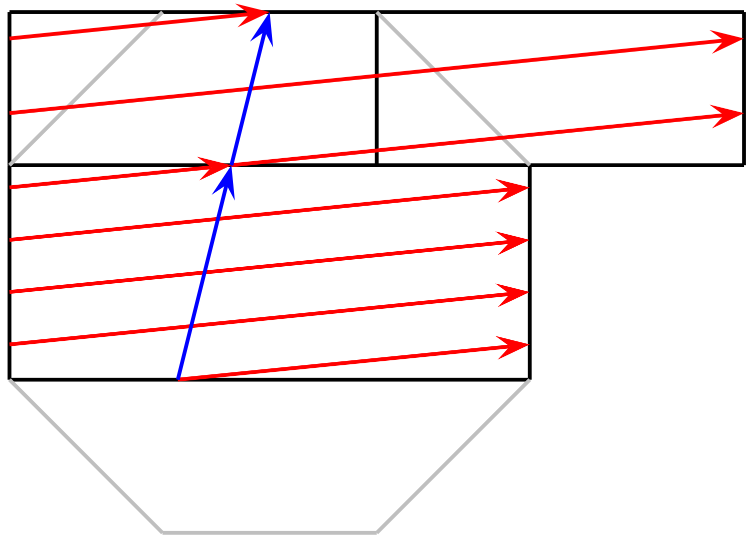


$$\text{normalized horizontal street length} = \frac{\text{length of horizontal street}}{\text{width of horizontal street}}$$



normalized horizontal street lengths $2(1+\sqrt{2})$ and $1+\sqrt{2}$

street-rational polyrectangle surface

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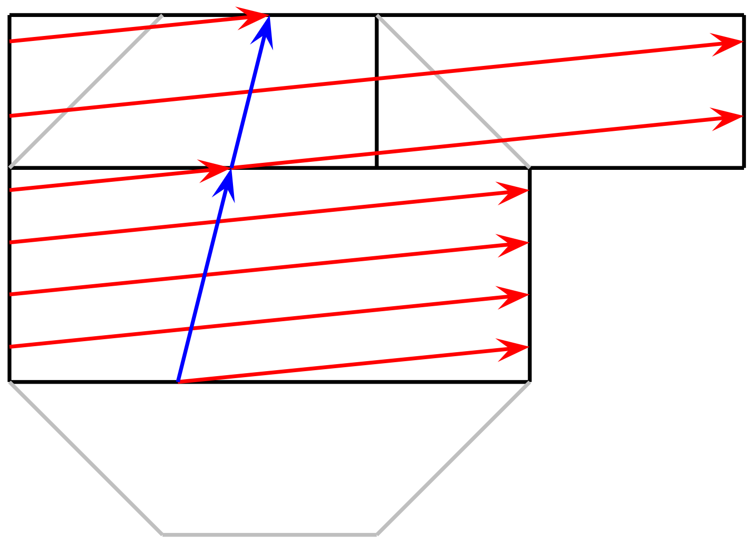
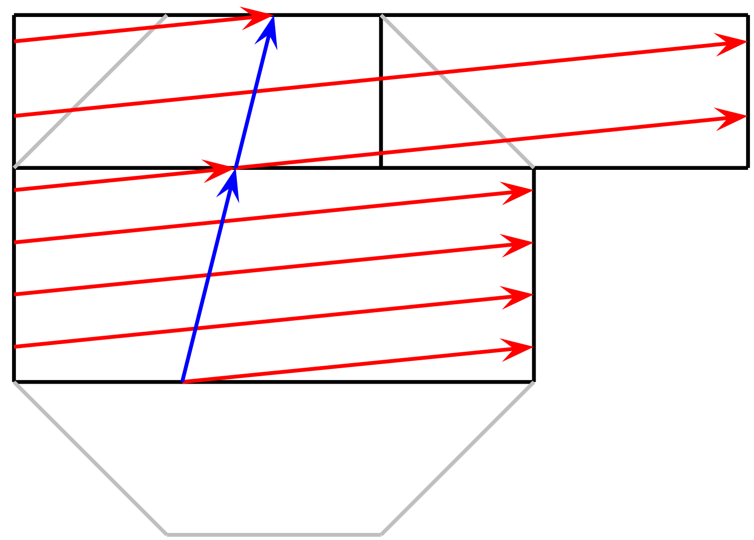


normalized horizontal street lengths $2(1+\sqrt{2})$ and $1+\sqrt{2}$

h^* – normalized horizontal street-LCM

= smallest integer multiple of all normalized horizontal street lengths normalized horizontal street length $= \dfrac{\text{length of horizontal street}}{\text{width of horizontal street}}$



normalized horizontal street lengths $2(1+\sqrt{2})$ and $1+\sqrt{2}$

h^* − normalized horizontal street-LCM $\qquad h^* = 2(1+\sqrt{2})$

= smallest integer multiple of all normalized horizontal street lengths

street-rational polyrectangle surface

h^* – normalized horizontal street-LCM

v^* – normalized vertical street-LCM

street-rational polyrectangle surface

h^* – normalized horizontal street-LCM

v^* – normalized vertical street-LCM

if start with almost vertical geodesic

slope $\alpha = v^* a_0 + \frac{1}{h^* a_1 + \frac{1}{v^* a_2 + \frac{1}{h^* a_3 + \cdots}}}$ with $a_0, a_1, a_2, a_3, \ldots \in \mathbb{N}$

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – finite street-rational polyrectangle surface

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infinitely many explicitly given slopes α

$\mathcal{L}$ – 1-direction geodesic in $\mathcal{P}$ with slope α

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superdensity of $\mathcal{L}$

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can compute irregularity exponent

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Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – regular k-gon surface for even $k \geqslant 8$

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Beck–C–Yang ($\geqslant$ 2020)

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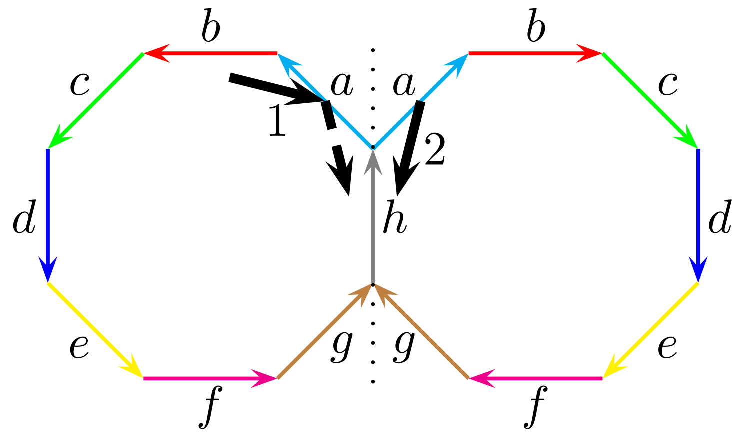
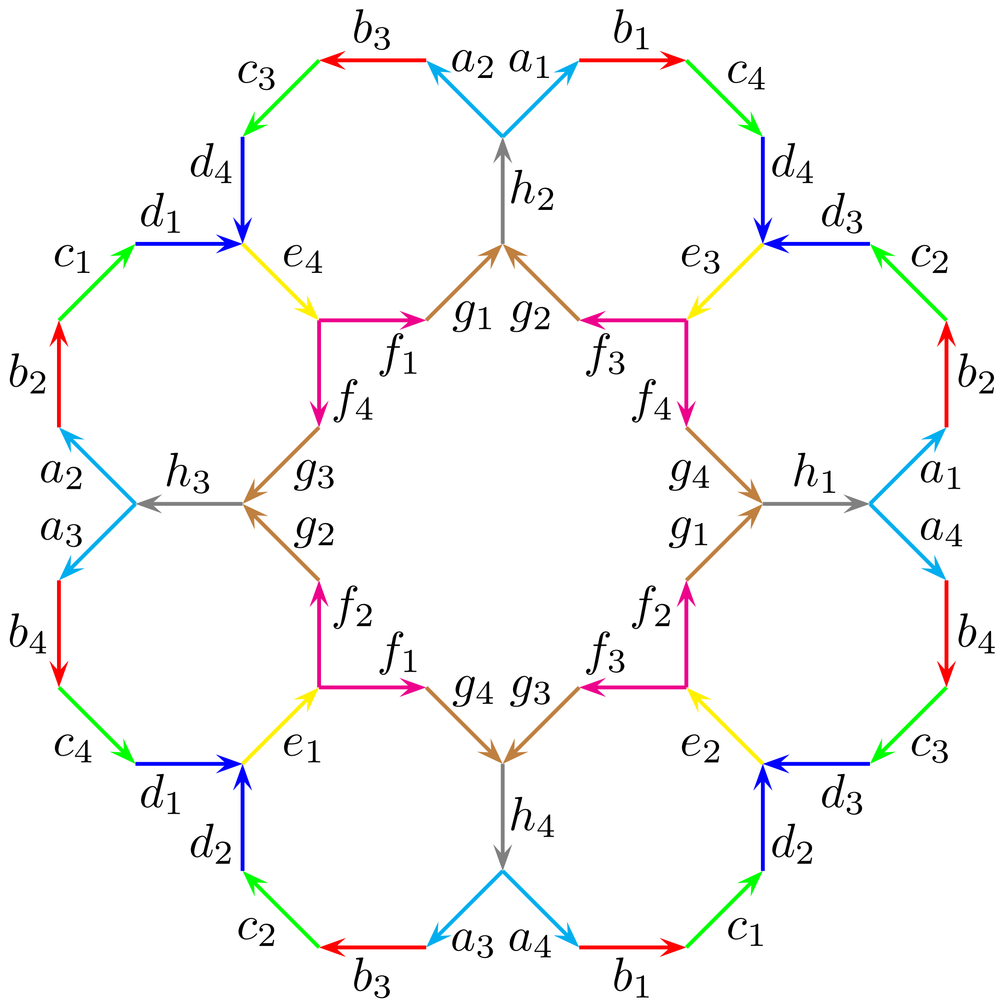
? $k = 6$?

billiard in regular octagon region

partial unfolding of billiard in left regular octagon

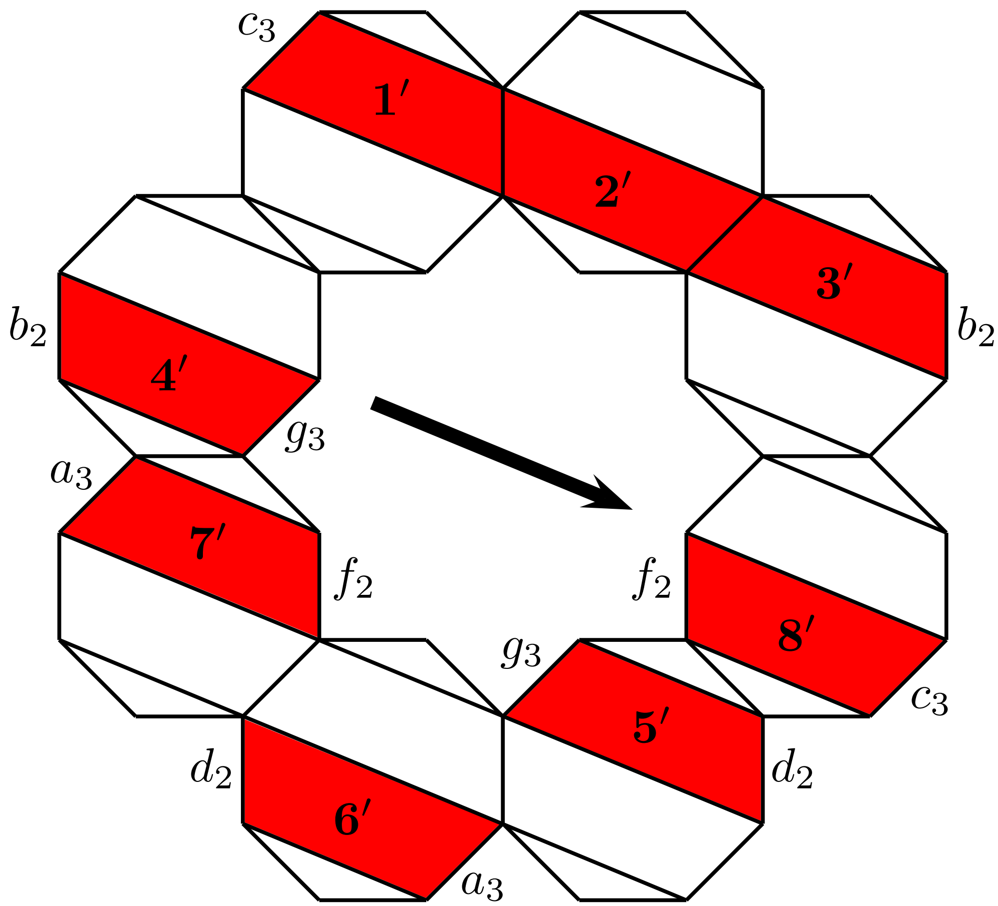


$\hookrightarrow$ 1-direction geodesic on surface



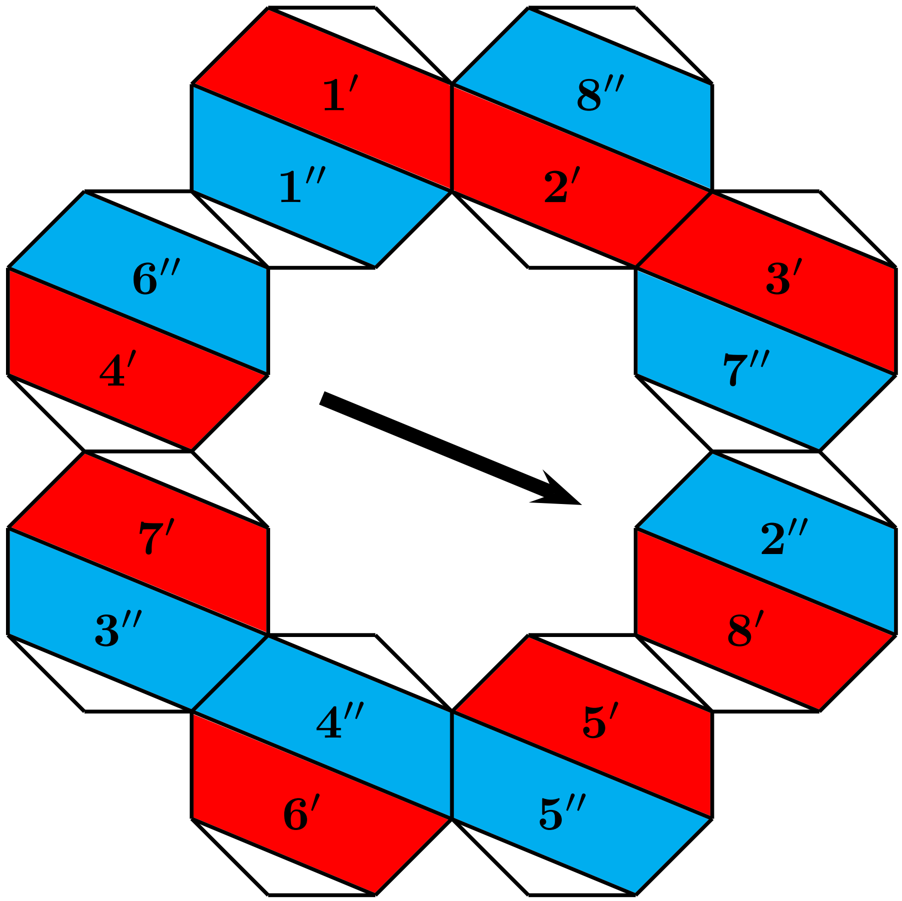
billiard in regular octagon region

$\hookrightarrow$ 1-direction geodesic on surface



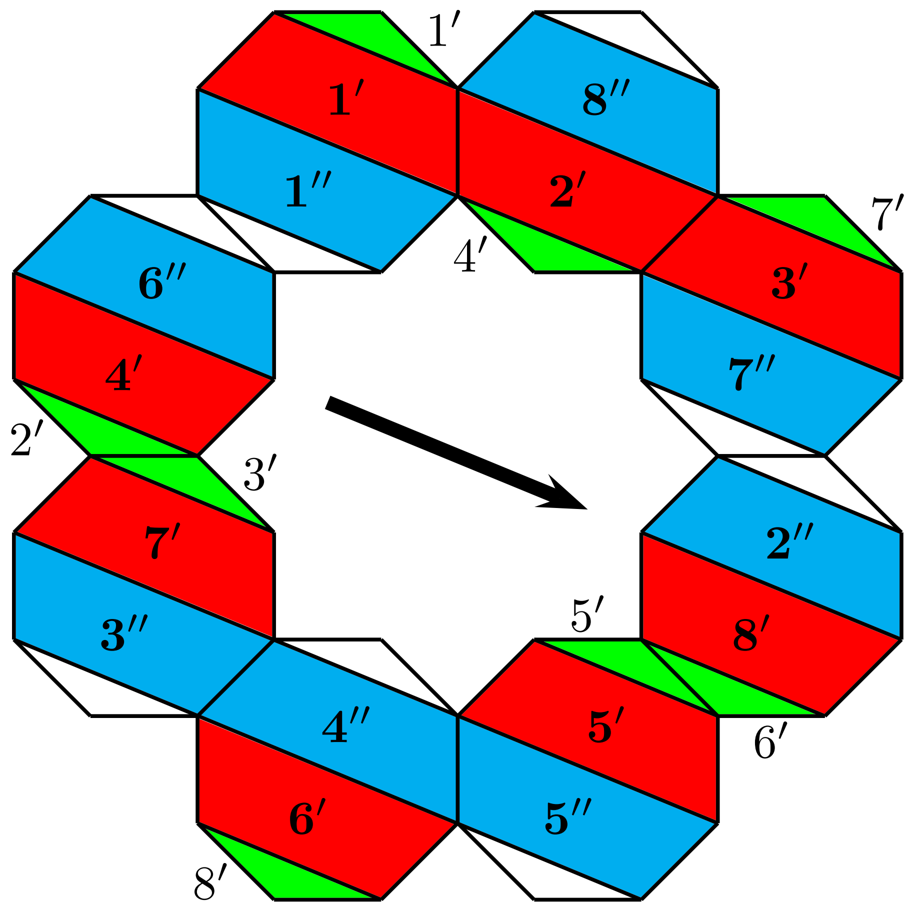
billiard in regular octagon region

↪ 1-direction geodesic on surface



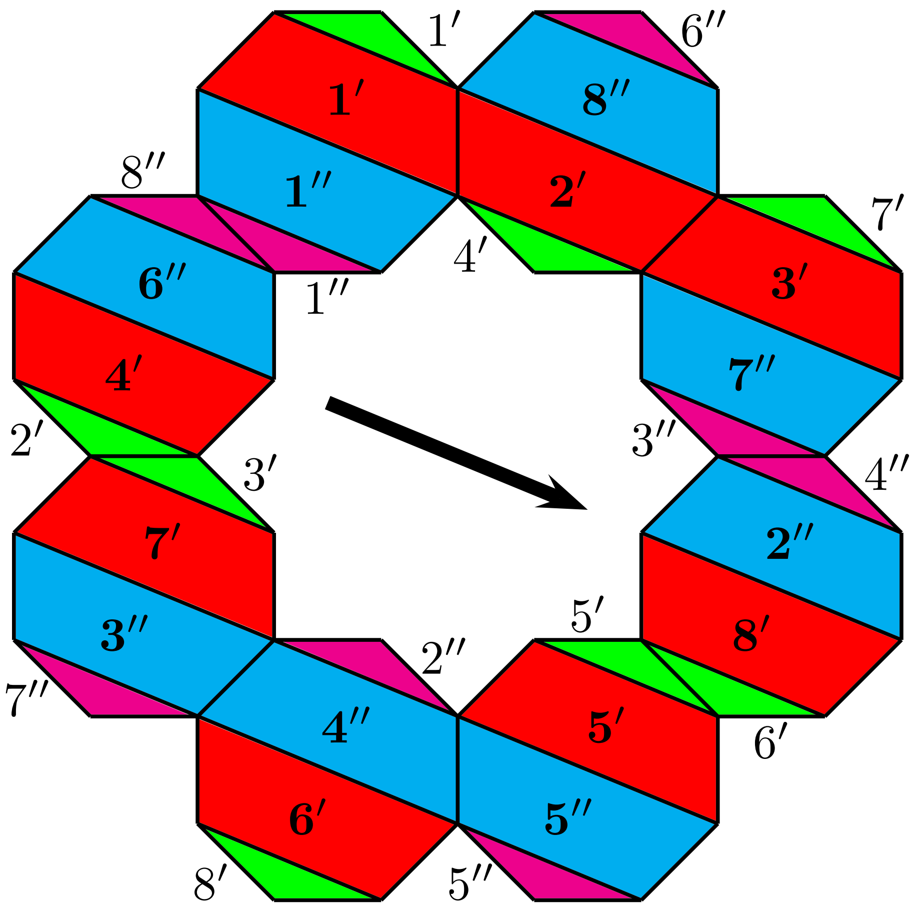
billiard in regular octagon region

↪ 1-direction geodesic on surface



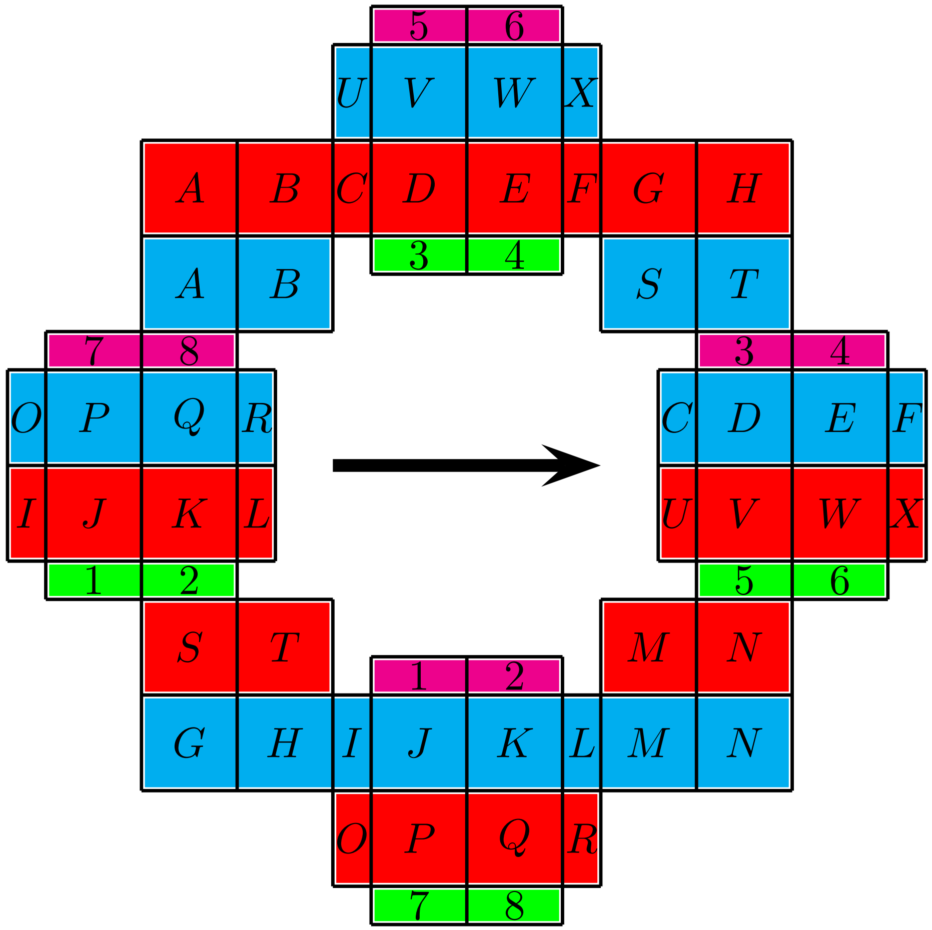
billiard in regular octagon region

$\hookrightarrow$ 1-direction geodesic on surface



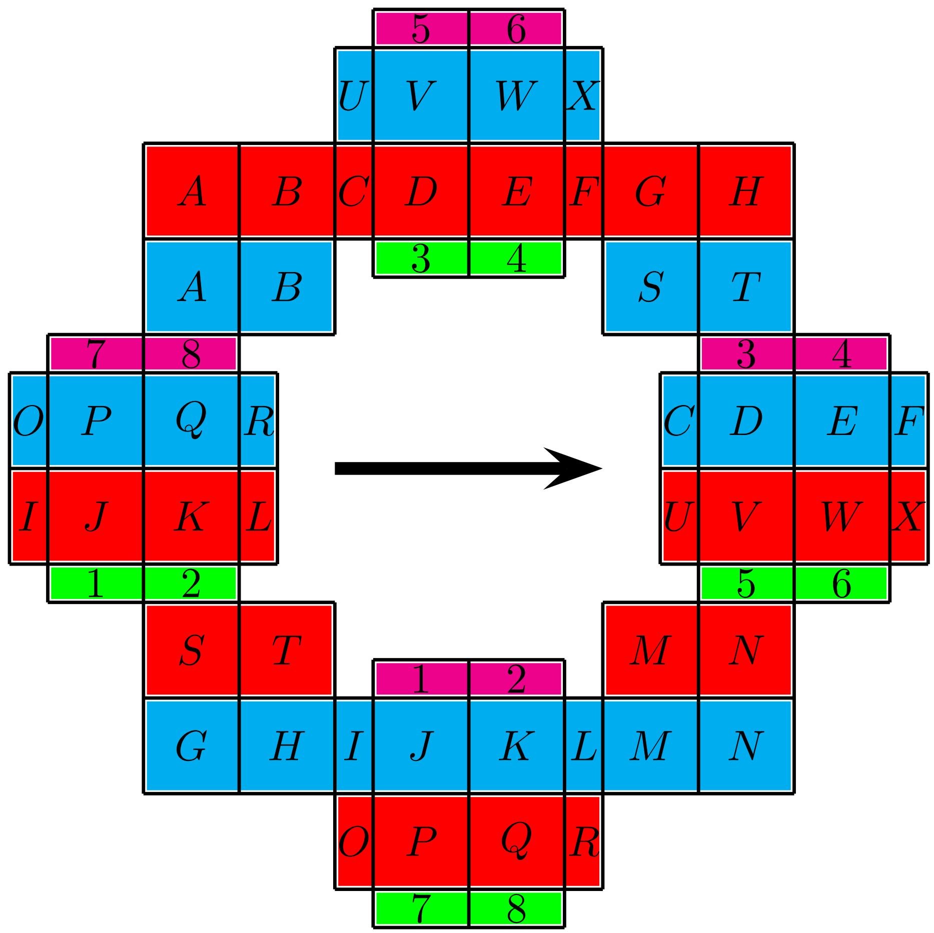
billiard in regular octagon region

$\hookrightarrow$ 1-direction geodesic on surface



billiard in regular octagon region

↪ 1-direction geodesic in street-rational polyrectangle surface



Beck–C–Yang ($\geqslant 2020$)

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infinitely many explicitly given slopes α

$\mathcal{L}$ – billiard orbit in $\mathcal{P}$ with initial slope α

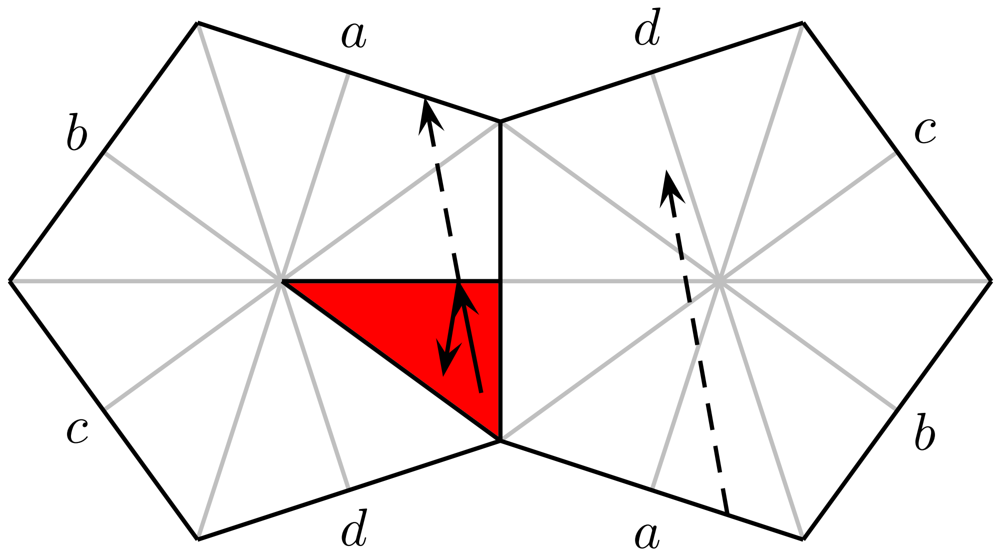
superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

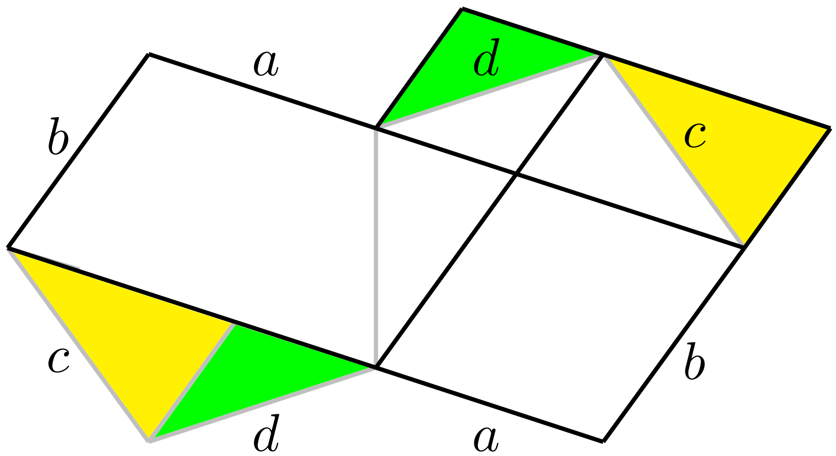
billiard in right triangle with angle $\pi/5$

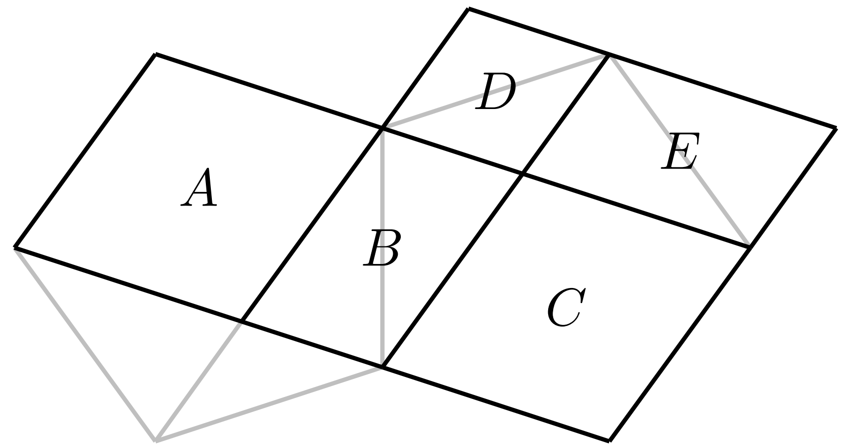
billiard in right triangle with angle $\pi/5$



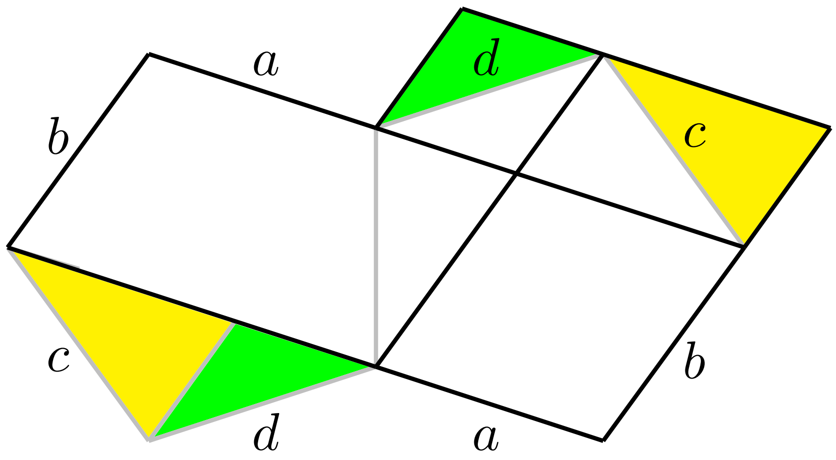
$\hookrightarrow$ 1-direction geodesic on regular double-pentagon surface

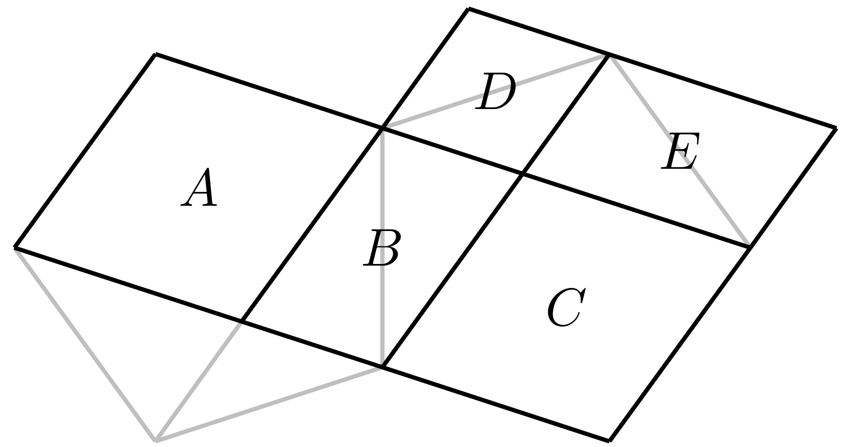
1-direction geodesic on regular double-pentagon surface


a
d
b
c
c
b
d
a


D
E
A
B
C

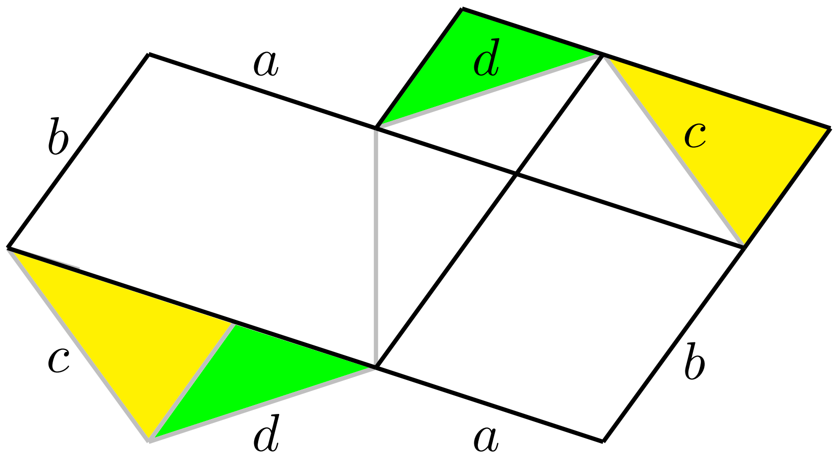
1-direction geodesic on regular double-pentagon surface

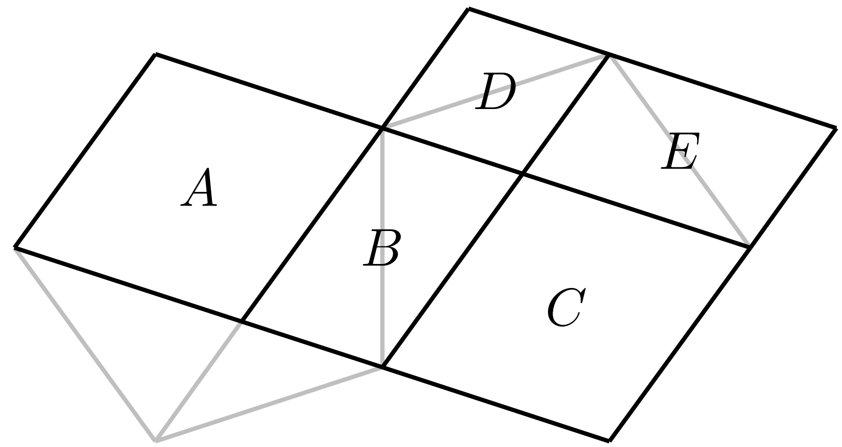




$\hookrightarrow$ 1-direction geodesic on street-rational polyparallelogram surface

1-direction geodesic on regular double-pentagon surface





$\hookrightarrow$ 1-direction geodesic on street-rational polyparallelogram surface

visualized as a street-rational polyrectangle surface

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ − regular double-k-gon surface for odd $k \geqslant 5$

infinitely many explicitly given slopes α

$\mathcal{L}$ − 1-direction geodesic in $\mathcal{P}$ with slope α

superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – right triangle with angle π/k for odd $k \geqslant 5$

infinitely many explicitly given slopes α

$\mathcal{L}$ – billiard orbit in $\mathcal{P}$ with initial slope α

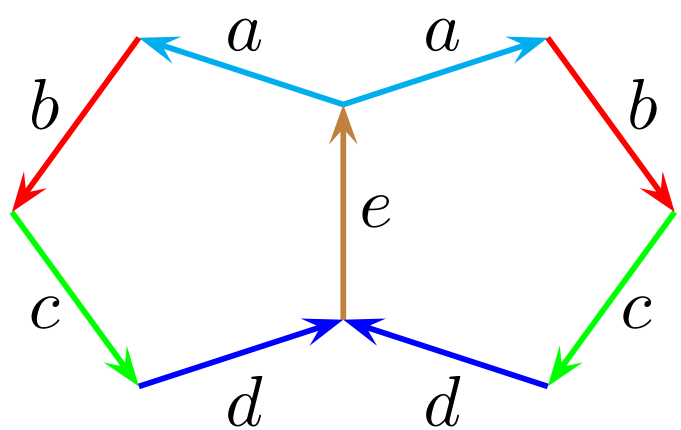
superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

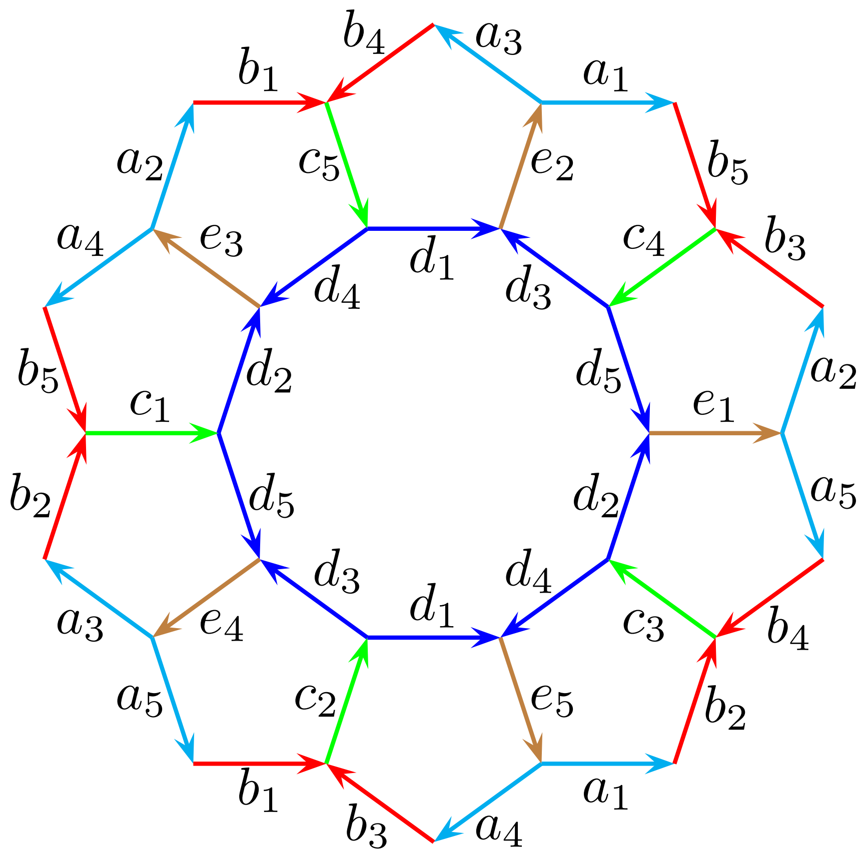
billiard in regular pentagon region

partial unfolding of billiard in left regular pentagon



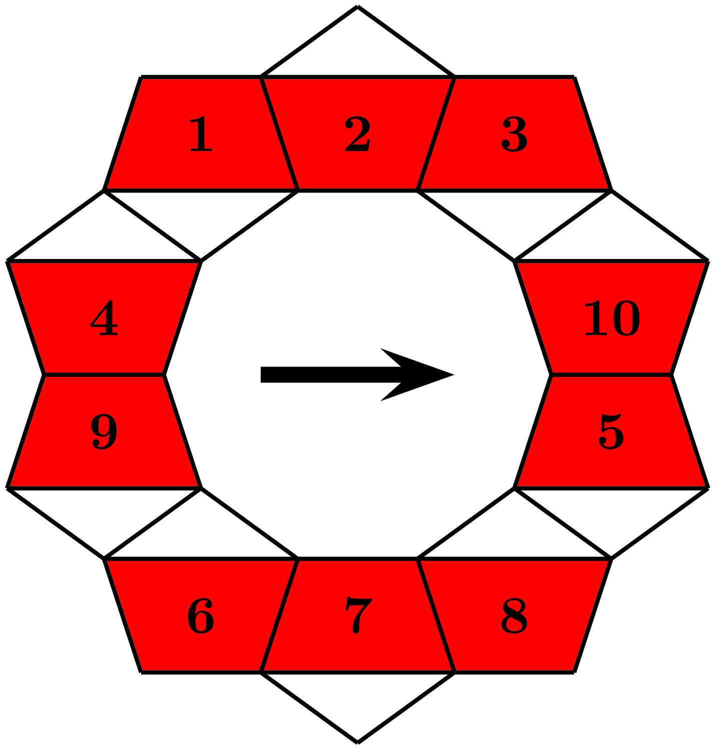
billiard in regular pentagon region

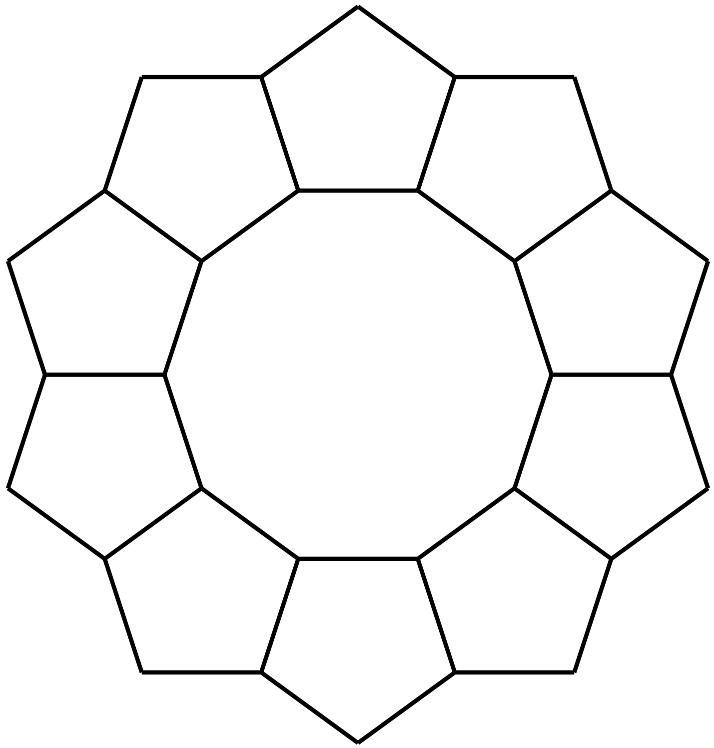
$\hookrightarrow$ 1-direction geodesic on surface



billiard in regular pentagon region

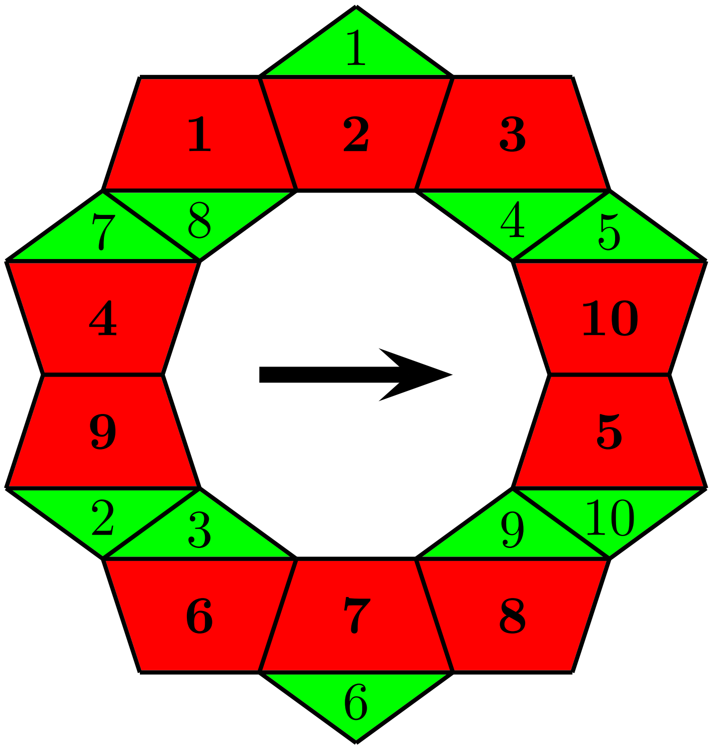
$\hookrightarrow$ 1-direction geodesic on surface

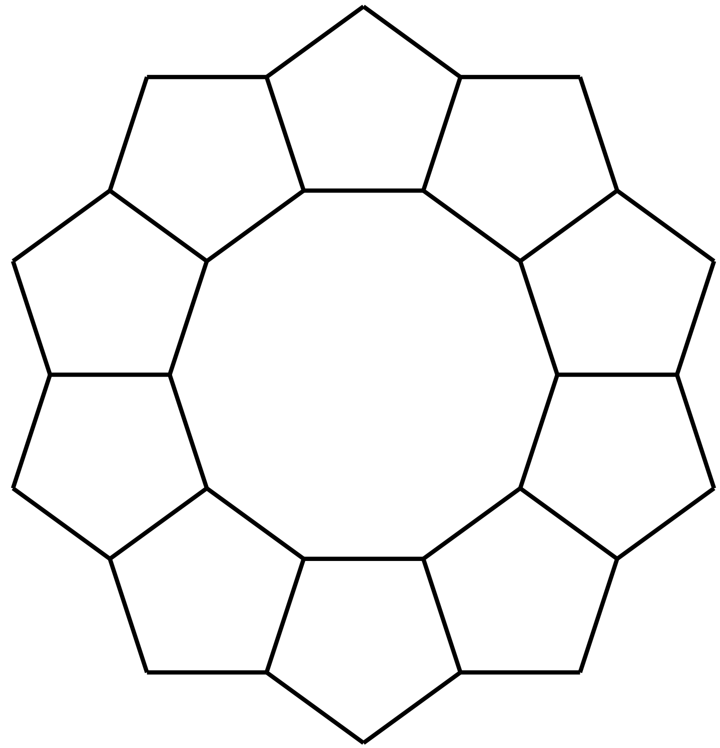




billiard in regular pentagon region

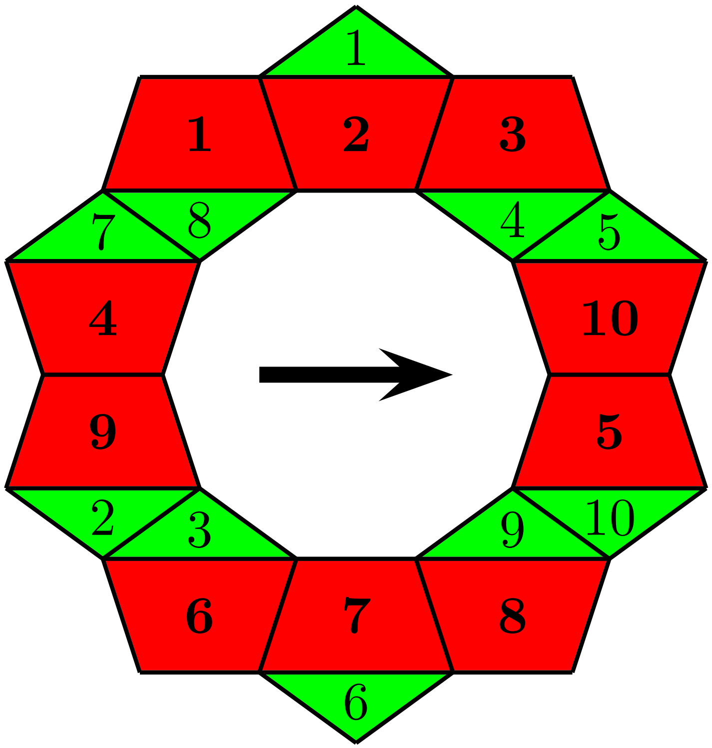
↪ 1-direction geodesic on surface

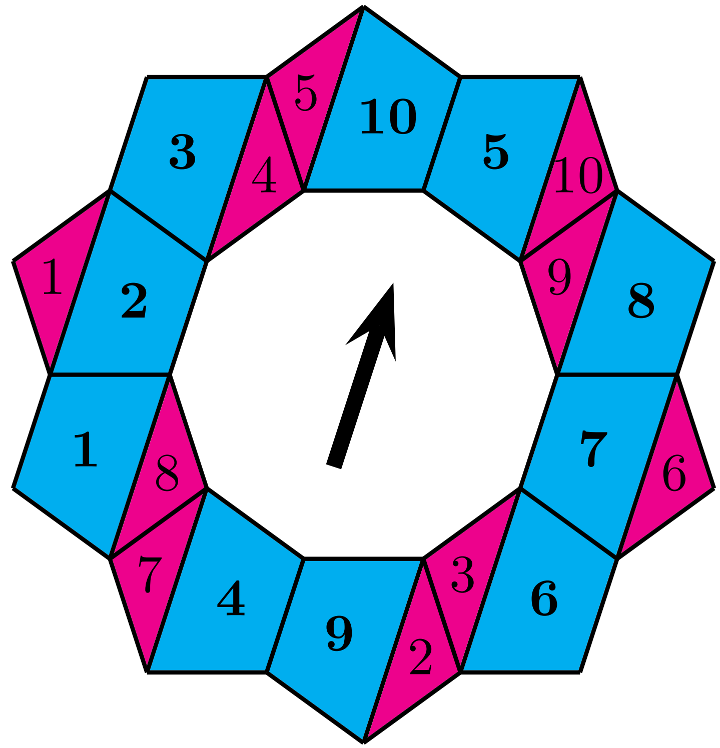




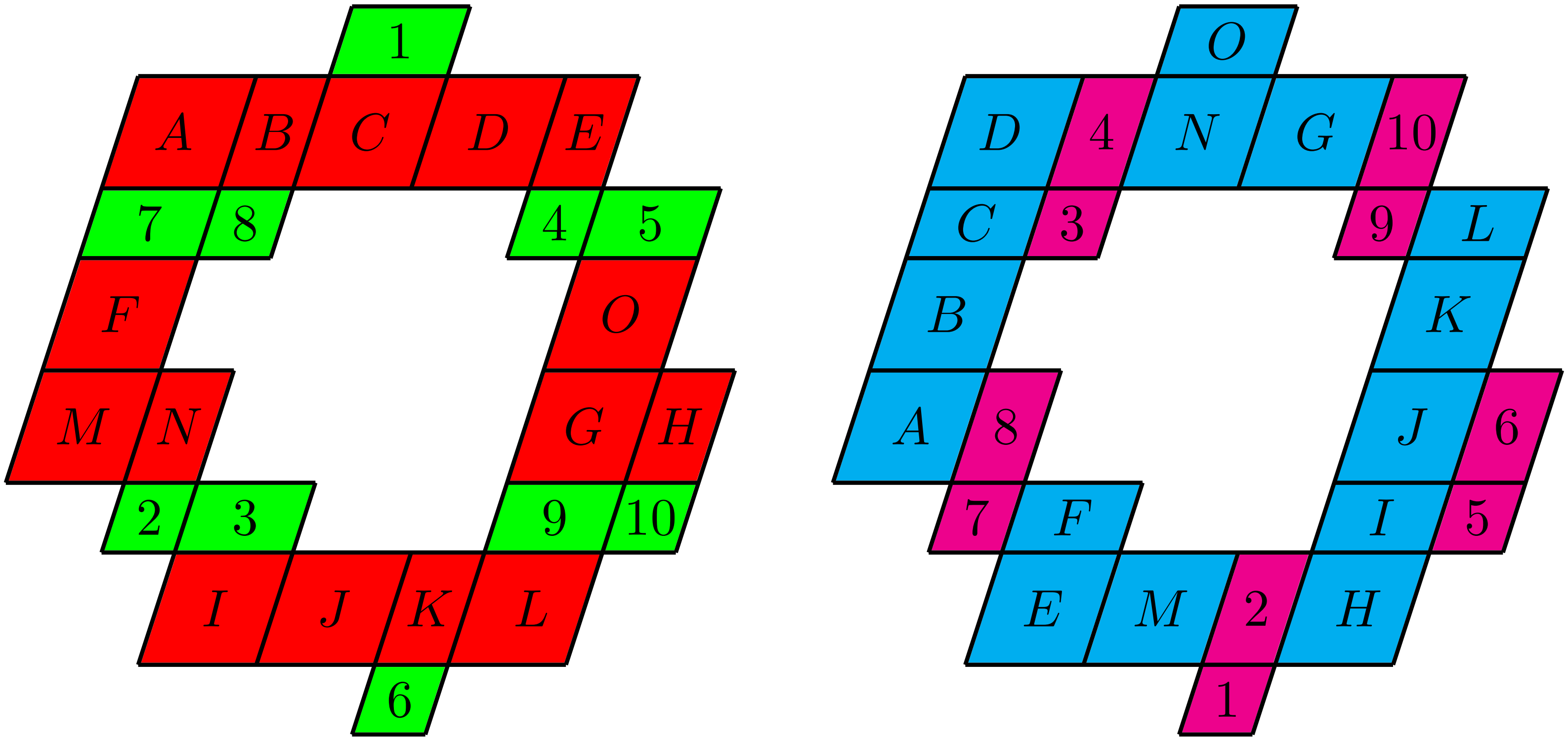
billiard in regular pentagon region

↪ 1-direction geodesic on surface

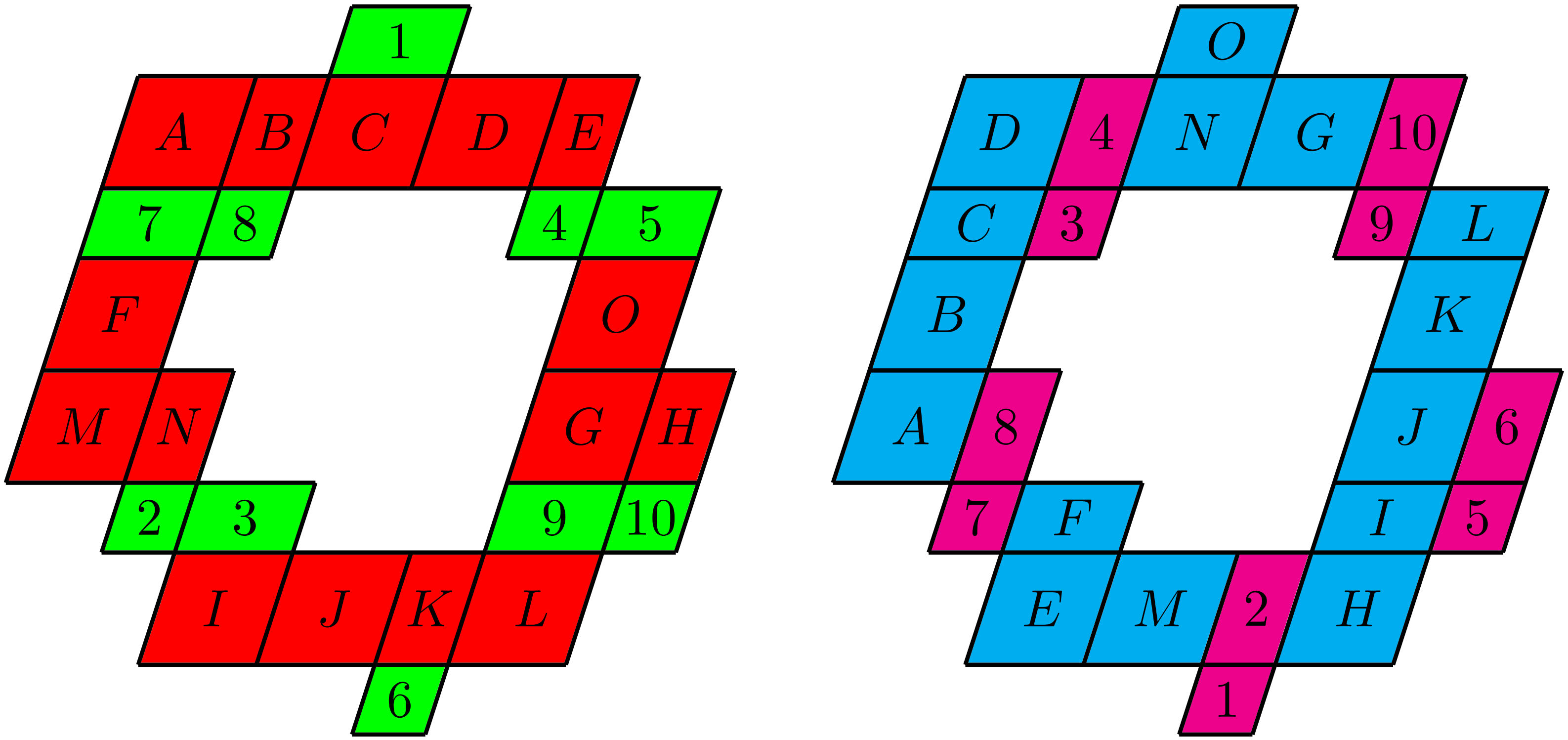




↪ 1-direction geodesic on surface



$\hookrightarrow$ 1-direction geodesic on street-rational polyparallelogram surface



Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – regular k-gon for odd $k \geqslant 5$

infinitely many explicitly given slopes α

$\mathcal{L}$ – billiard orbit in $\mathcal{P}$ with initial slope α

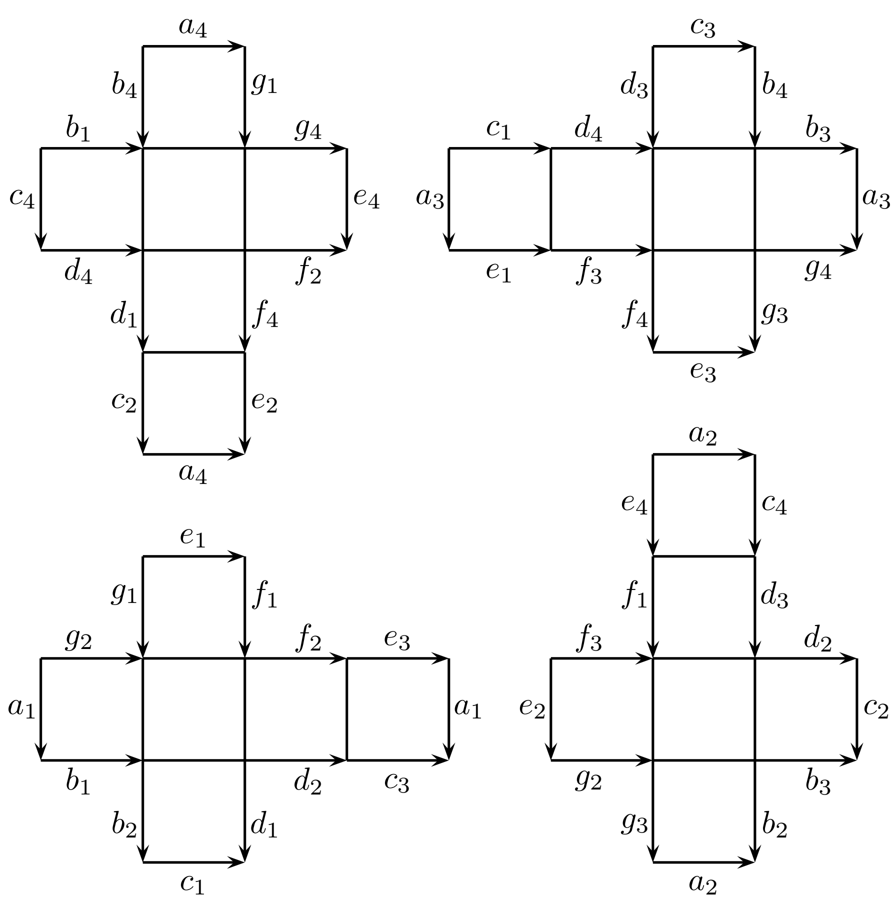
superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

geodesic on regular tetrahedron surface – integrable geodesic on regular tetrahedron surface – integrable

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on polysquare surface



geodesics on surfaces of the Platonic solids

geodesic on regular tetrahedron surface – integrable

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on polysquare surface

geodesic on regular dodecahedron surface

geodesics on surfaces of the Platonic solids

geodesic on regular tetrahedron surface – integrable

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on polysquare surface

geodesic on regular dodecahedron surface

standard net of regular dodecahedron surface has 12 regular pentagons

geodesics on surfaces of the Platonic solids

geodesic on regular tetrahedron surface – integrable

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on polysquare surface

geodesic on regular dodecahedron surface

$\hookrightarrow$ 1-direction geodesic on surface with 120 regular pentagon faces

geodesics on surfaces of the Platonic solids

geodesic on regular tetrahedron surface – integrable

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on polysquare surface

geodesic on regular dodecahedron surface

$\hookrightarrow$ 1-direction geodesic on surface with 120 regular pentagon faces

1-direction geodesic on finite street-rational polyparallelogram surface

geodesics on surfaces of the Platonic solids

geodesic on regular tetrahedron surface – integrable

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on polysquare surface

geodesic on regular dodecahedron surface

$\hookrightarrow$ 1-direction geodesic on surface with 120 regular pentagon faces

1-direction geodesic on finite street-rational polyparallelogram surface

geodesic on regular octahedron surface

geodesic on regular icosahedron surface

geodesics on surfaces of the Platonic solids

geodesic on regular tetrahedron surface – integrable

geodesic on cube surface $\hookrightarrow$ 1-direction geodesic on polysquare surface

geodesic on regular dodecahedron surface

$\hookrightarrow$ 1-direction geodesic on surface with 120 regular pentagon faces

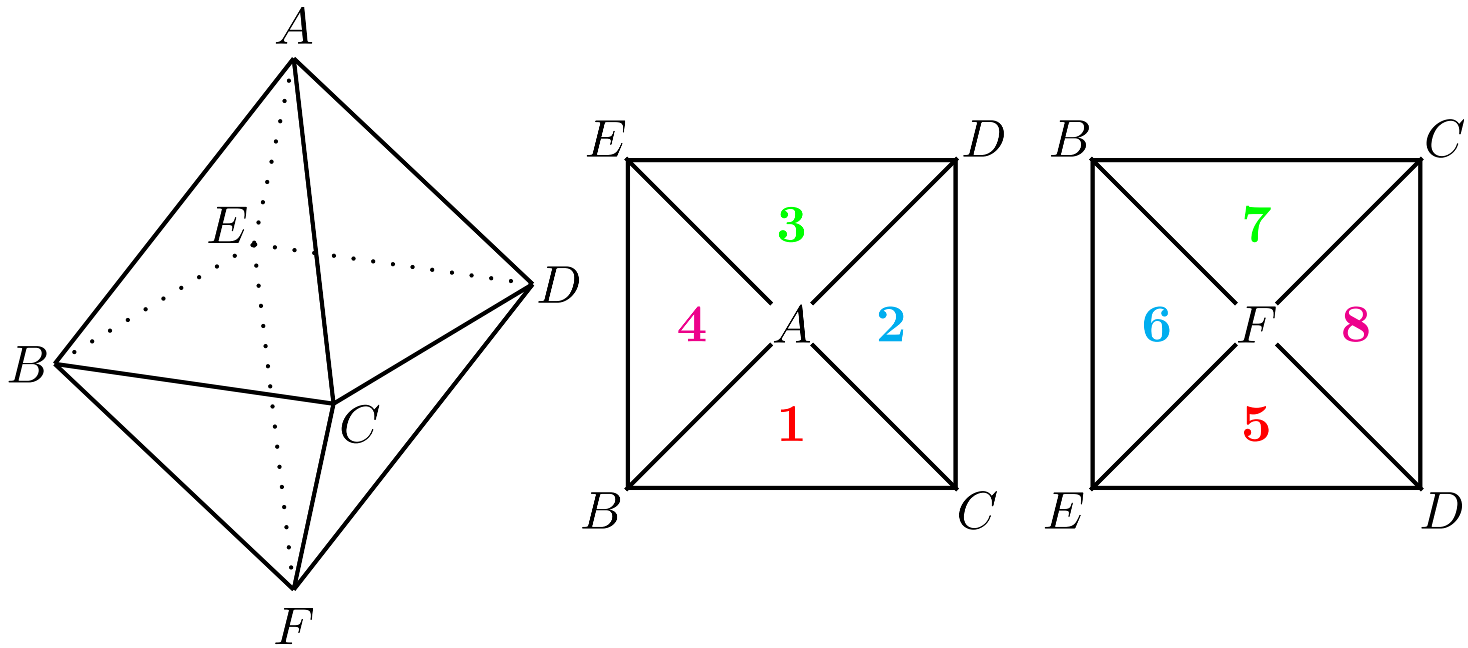
1-direction geodesic on finite street-rational polyparallelogram surface

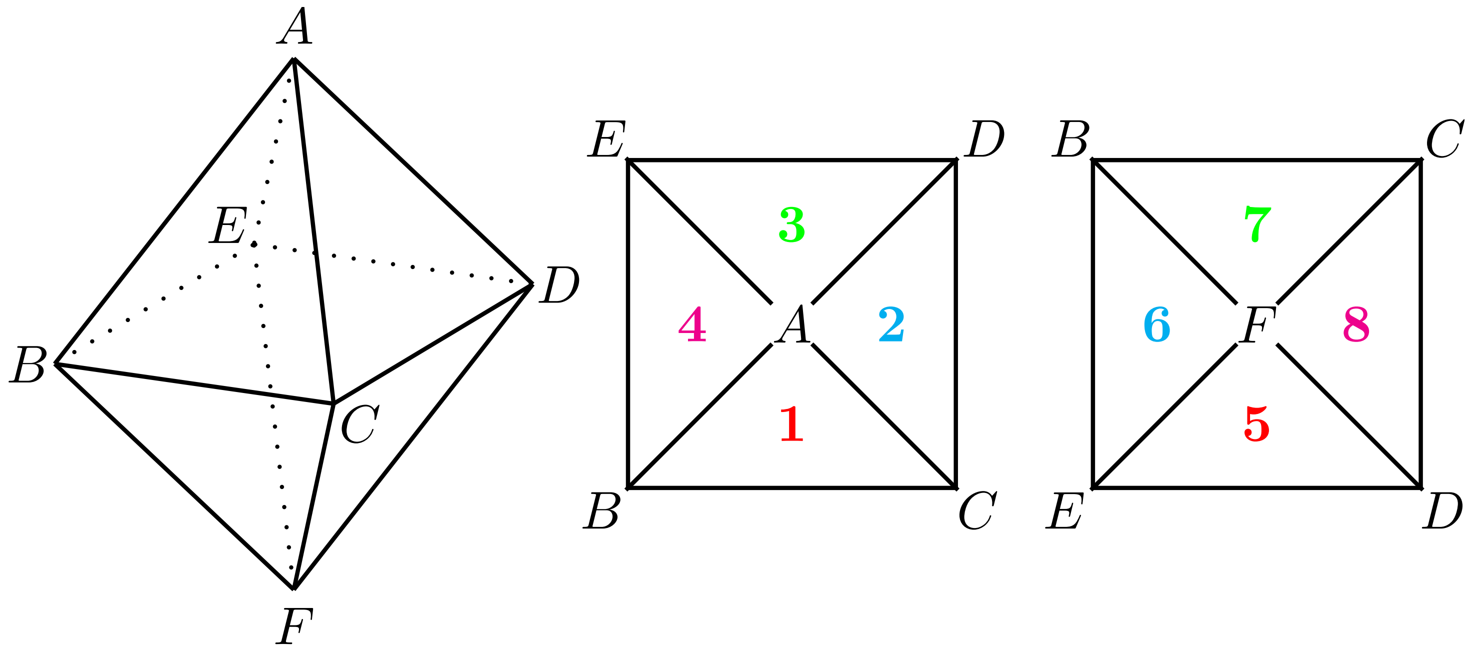
geodesic on regular octahedron surface

geodesic on regular icosahedron surface

polytriangle surfaces

geodesic on regular octahedron surface

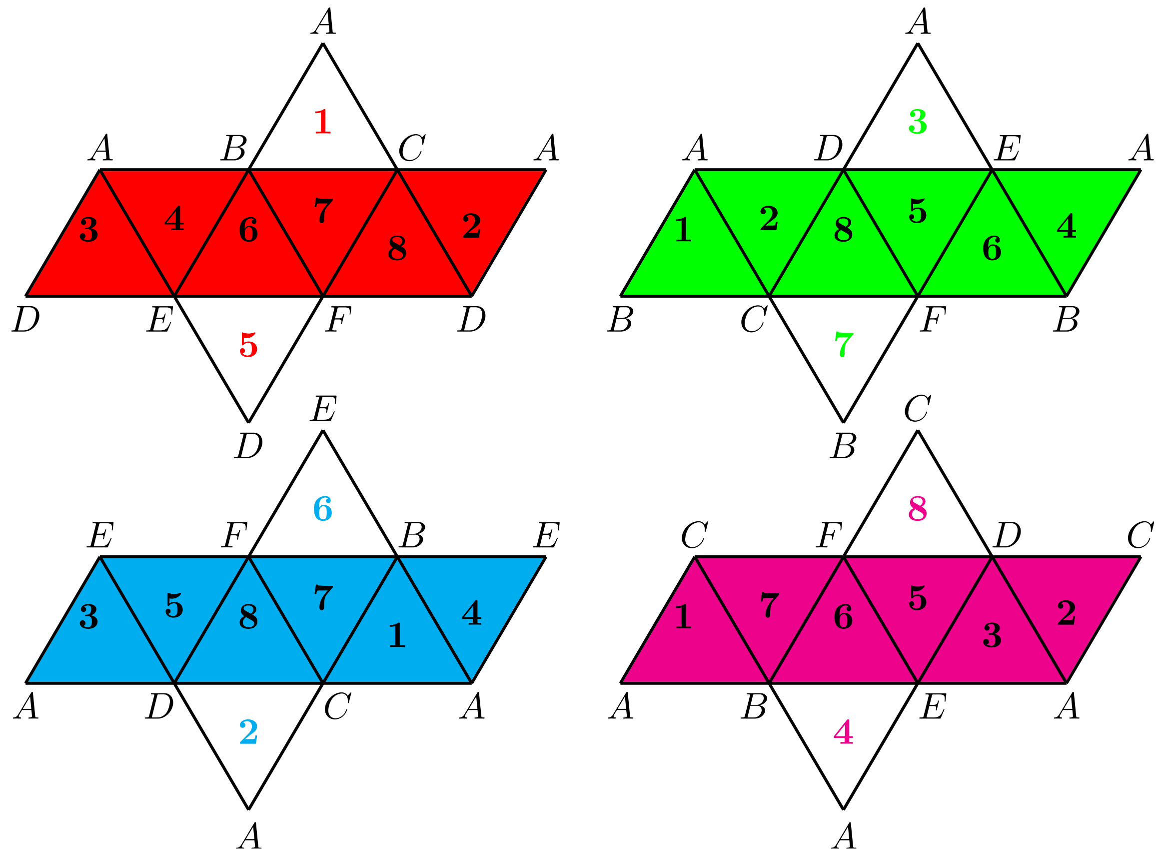




vertex-disjoint faces – (1, 5), (2, 6), (3, 7), (4, 8)

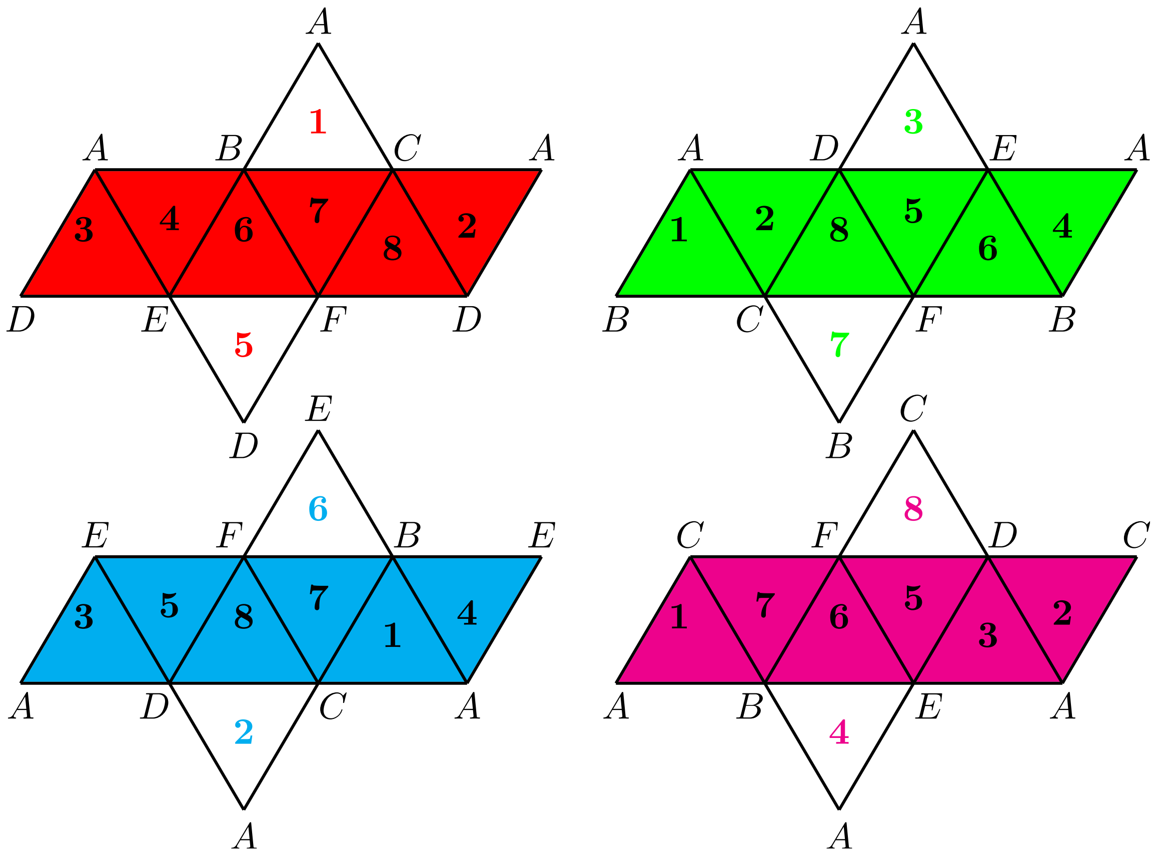
geodesic on regular octahedron surface

streets between vertex-disjoint faces – (1, 5), (2, 6), (3, 7), (4, 8)



geodesic on regular octahedron surface

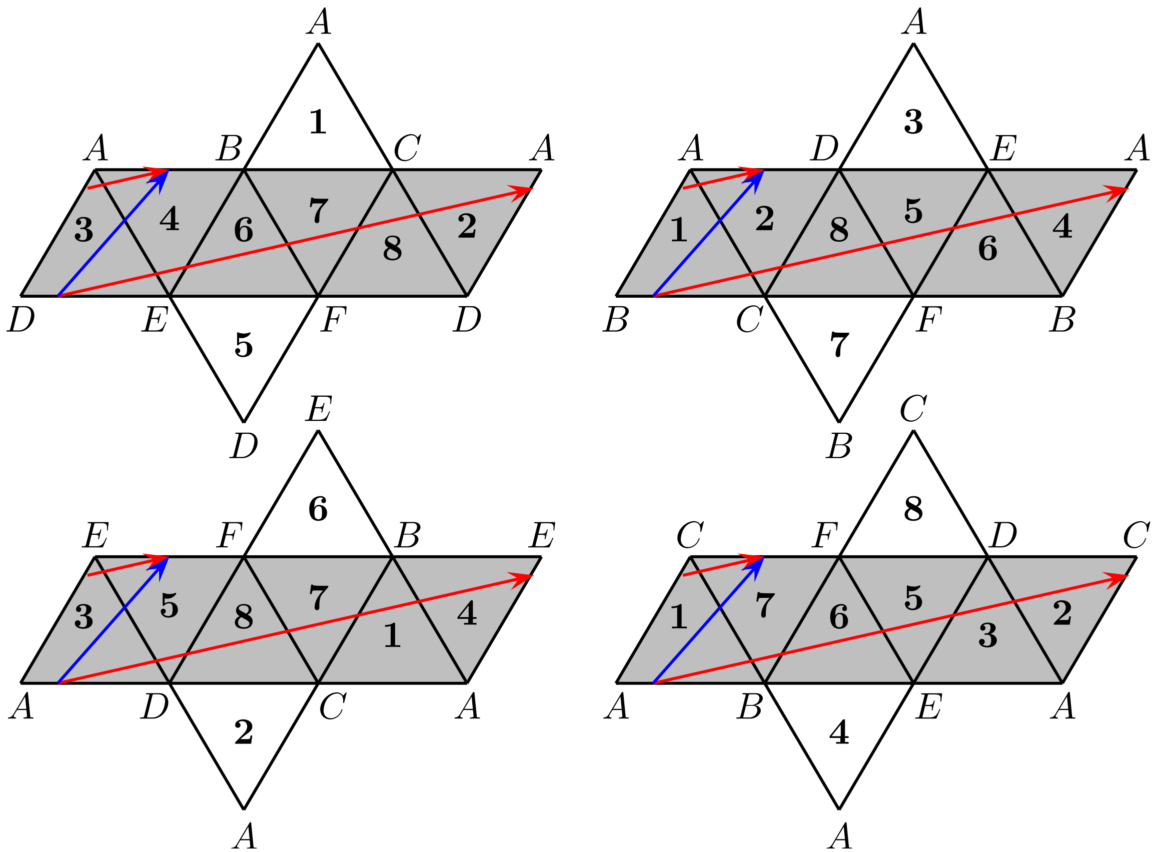
streets between vertex-disjoint faces – (1, 5), (2, 6), (3, 7), (4, 8)



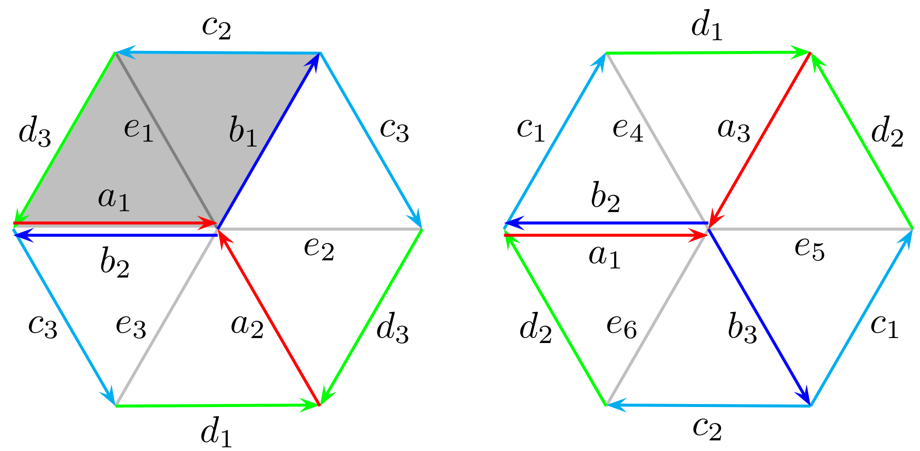
street-rational polyparallelogram surface

geodesic on regular octahedron surface

streets between vertex-disjoint faces – (1, 5), (2, 6), (3, 7), (4, 8)



detour crossings and shortcuts


c_2
d_3
e_1
b_1
c_3
a_1
b_2
e_2
c_3
e_3
a_2
d_3
d_1
d_1
c_1
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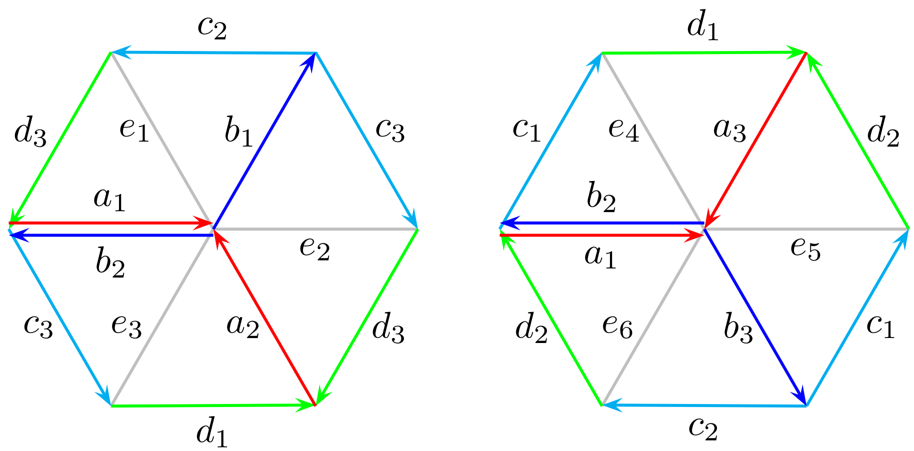
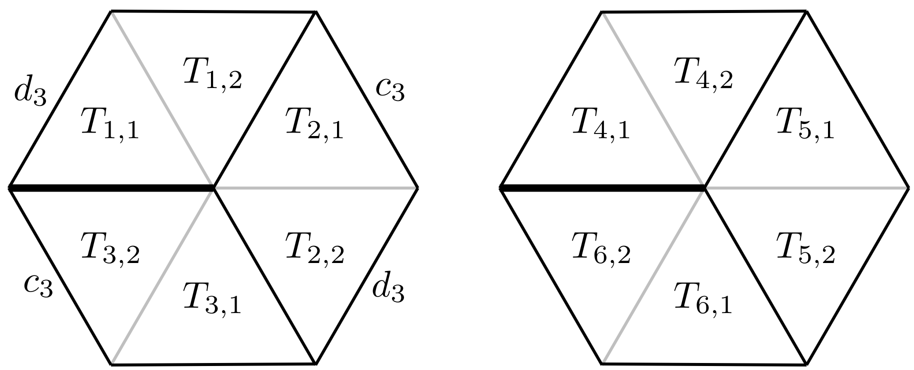
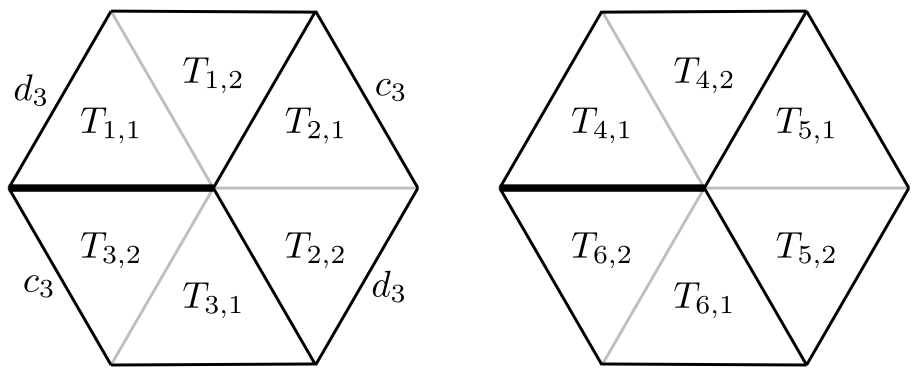
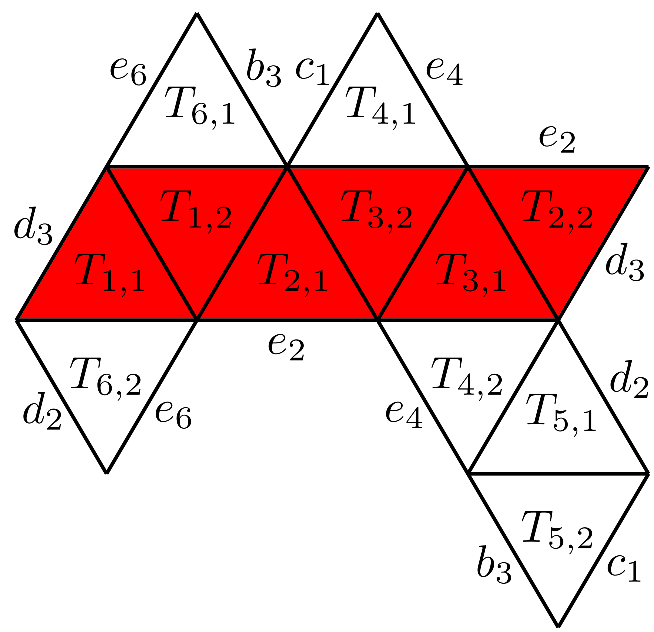
e_6
b_3
c_1
c_2

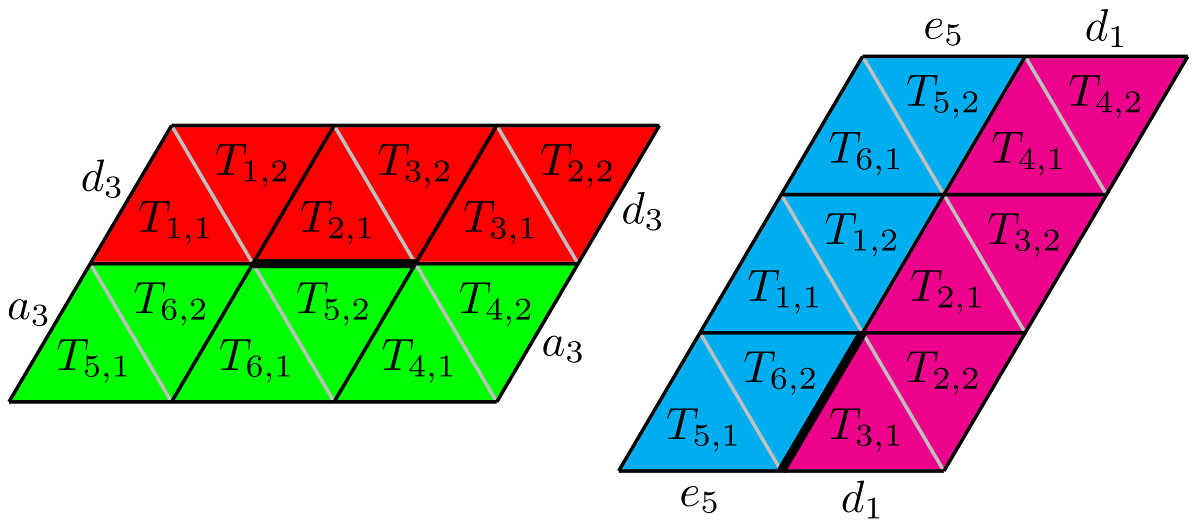

c_2
d_3
e_1
b_1
c_3
a_1
b_2
e_2
c_3
e_3
a_2
d_3
d_1
d_1
c_1
e_4
a_3
d_2
b_2
a_1
e_5
d_2
e_6
b_3
c_1
c_2

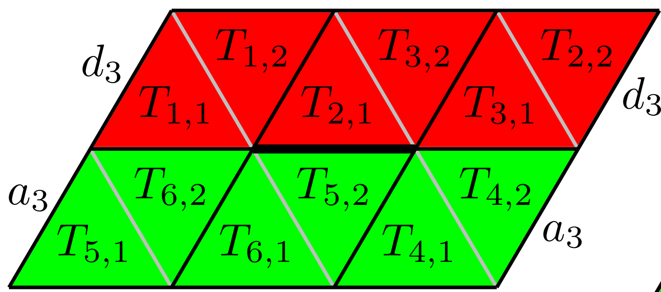
billiard on 60-degree rhombus

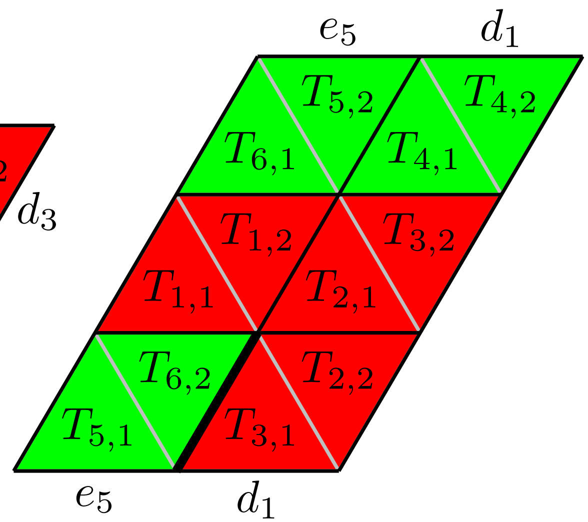



d_3
$T_{1,2}$
c_3
$T_{1,1}$
$T_{2,1}$
$T_{3,2}$
$T_{2,2}$
c_3
d_3
$T_{3,1}$
$T_{4,2}$
$T_{4,1}$
$T_{5,1}$
$T_{6,2}$
$T_{5,2}$
$T_{6,1}$


e_6
b_3
c_1
e_4
$T_{6,1}$
$T_{4,1}$
e_2
d_3
$T_{1,2}$
$T_{3,2}$
$T_{2,2}$
$T_{1,1}$
$T_{2,1}$
$T_{3,1}$
d_3
e_2
$T_{6,2}$
$T_{4,2}$
d_2
d_2
e_6
e_4
$T_{5,1}$
$T_{5,2}$
b_3
c_1


d_3
$T_{1,2}$
$T_{3,2}$
$T_{2,2}$
$T_{1,1}$
$T_{2,1}$
$T_{3,1}$
d_3
a_3
$T_{6,2}$
$T_{5,2}$
$T_{4,2}$
$T_{5,1}$
$T_{6,1}$
$T_{4,1}$
a_3
e_5
d_1
$T_{5,2}$
$T_{4,2}$
$T_{6,1}$
$T_{4,1}$
$T_{1,2}$
$T_{3,2}$
$T_{1,1}$
$T_{2,1}$
$T_{6,2}$
$T_{2,2}$
$T_{5,1}$
$T_{3,1}$
e_5
d_1


d_3
$T_{1,2}$
$T_{3,2}$
$T_{2,2}$
$T_{1,1}$
$T_{2,1}$
$T_{3,1}$
d_3
a_3
$T_{6,2}$
$T_{5,2}$
$T_{4,2}$
$T_{5,1}$
$T_{6,1}$
$T_{4,1}$
a_3


e_5
d_1
$T_{5,2}$
$T_{4,2}$
$T_{6,1}$
$T_{4,1}$
$T_{1,2}$
$T_{3,2}$
$T_{1,1}$
$T_{2,1}$
$T_{6,2}$
$T_{2,2}$
$T_{5,1}$
$T_{3,1}$
e_5
d_1

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – finite polytriangle surface

infinitely many explicitly given slopes α

$\mathcal{L}$ – geodesic in $\mathcal{P}$ with slope α

superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – finite polytriangle region

infinitely many explicitly given slopes α

$\mathcal{L}$ – billiard orbit in $\mathcal{P}$ with initial slope α

superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

shortline method works for all Veech surfaces

shortline method works for all Veech surfaces

Veech (1989)

$\mathcal{P}$ – Veech surface

– street-rational decomposition in any direction with rational slope

shortline method works for all Veech surfaces

Veech (1989)

$\mathcal{P}$ – Veech surface

– street-rational decomposition in any direction with rational slope

– 1-direction geodesics exhibit uniform-periodic dichotomy – optimal

shortline method works for all Veech surfaces

Veech (1989)

$\mathcal{P}$ – Veech surface

– street-rational decomposition in any direction with rational slope

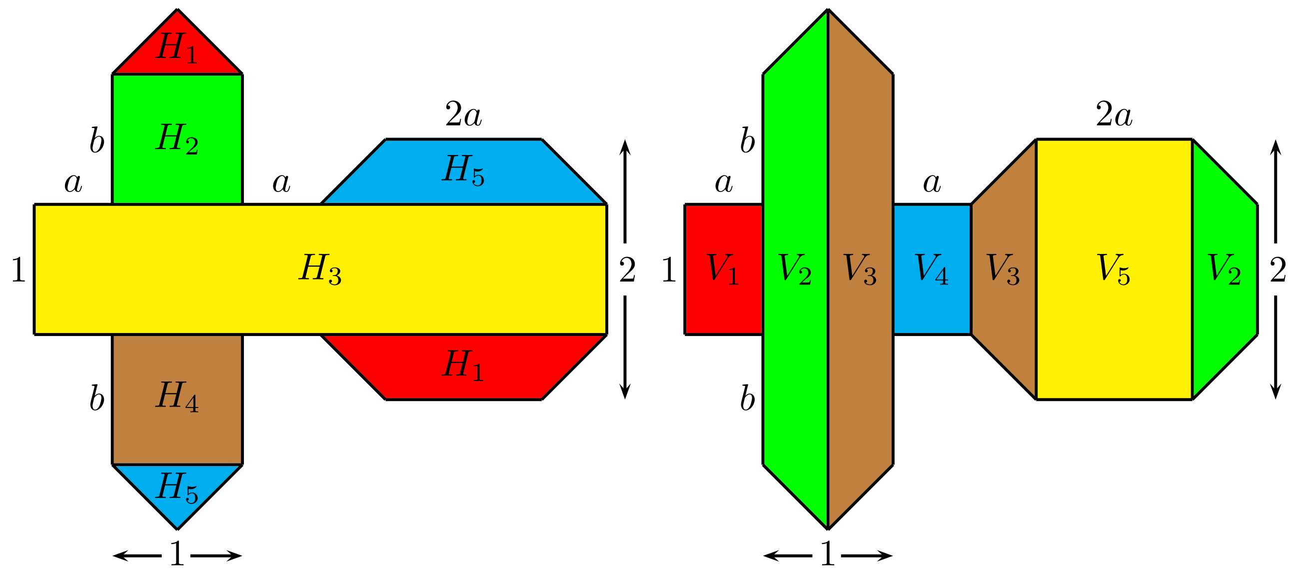
– 1-direction geodesics exhibit uniform-periodic dichotomy – optimal

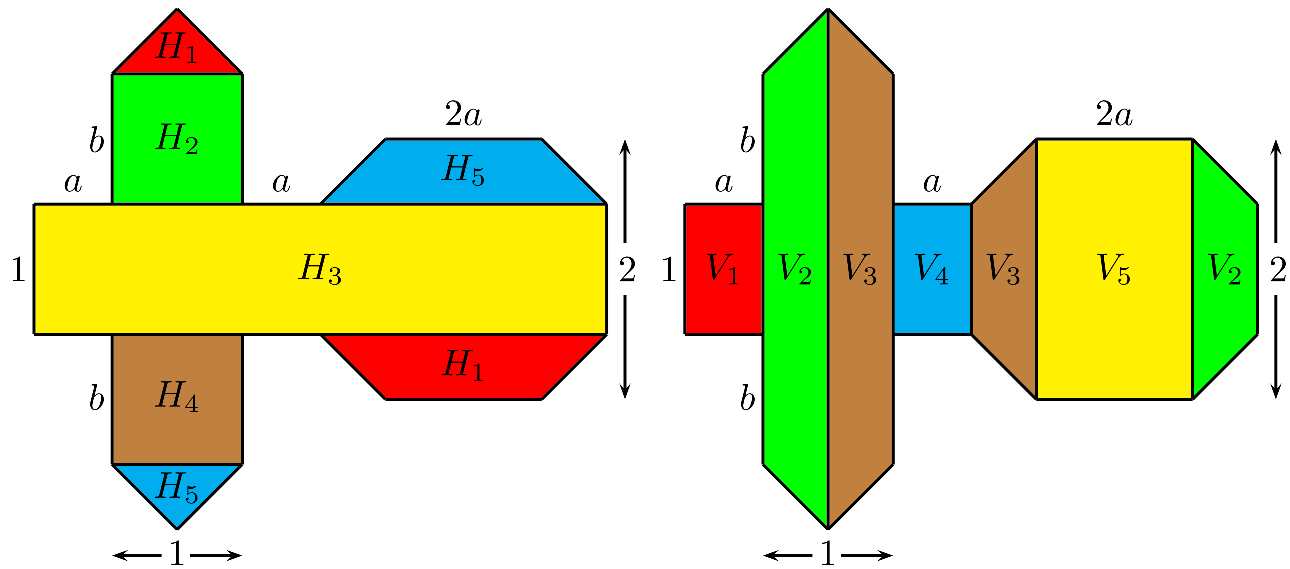
polysquare surfaces (including flat torus) and polytriangle surfaces

translation surfaces of regular polygon billiards

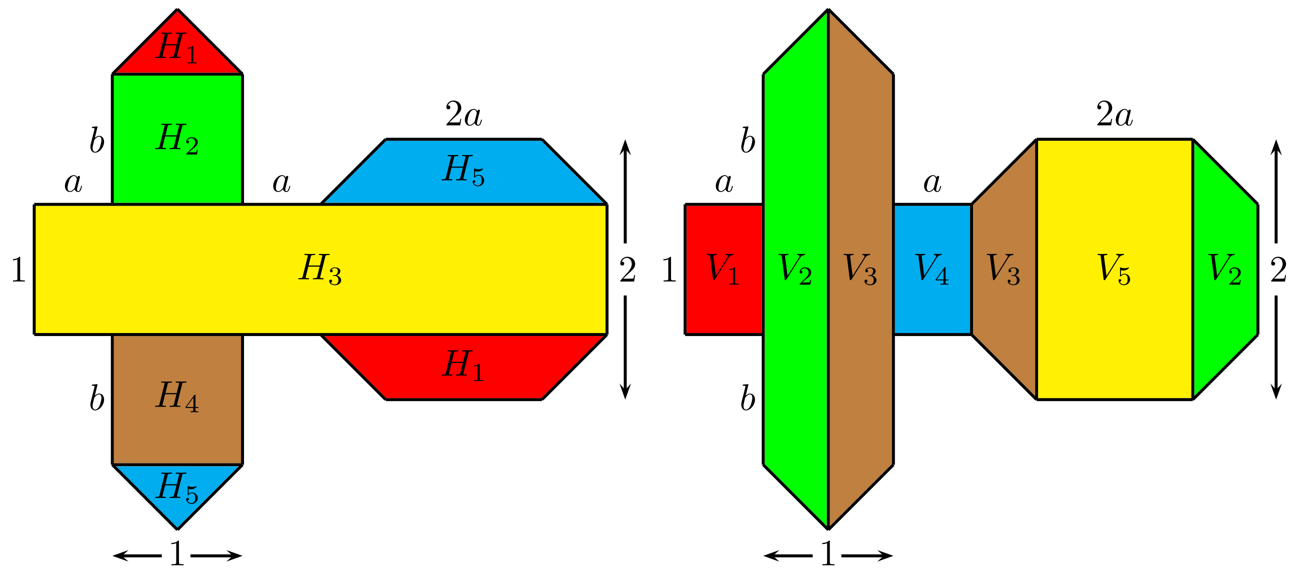
others

class of cathedral surfaces – McMullen–Mukamel–Wright (2017)



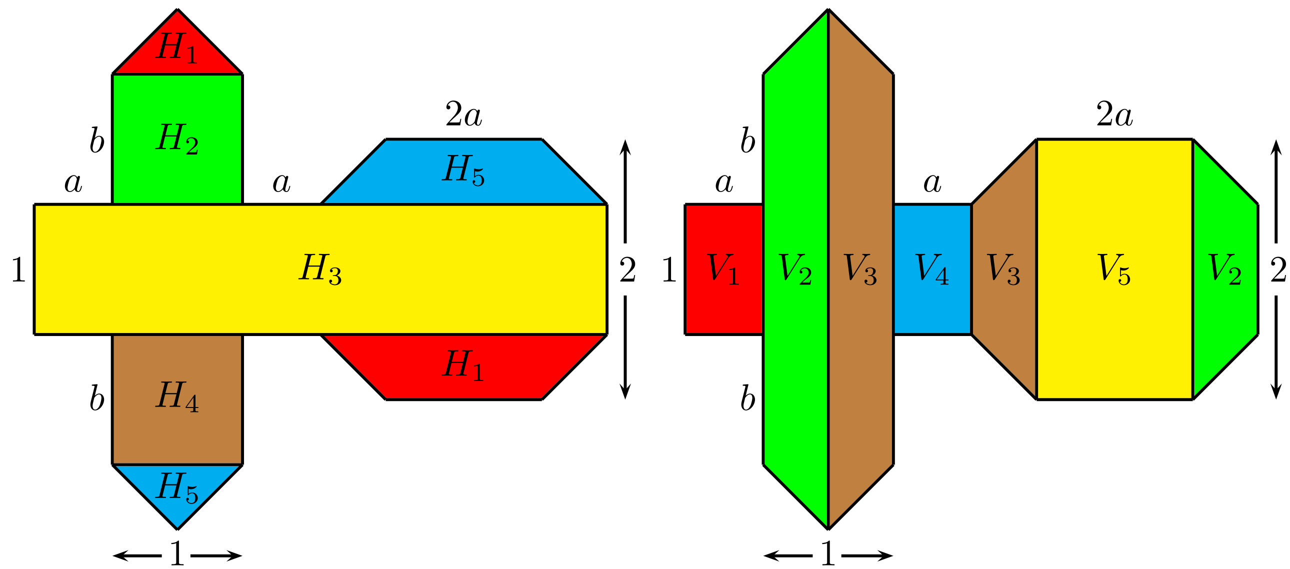


$\frac{\text{length}}{\text{width}}$ of horizontal streets $\quad \frac{1+2a}{1/2}, \ \frac{1}{b}, \ \frac{2+4a}{1}, \ \frac{1}{b}, \ \frac{1+2a}{1/2}$



$\frac{\text{length}}{\text{width}}$ of horizontal streets $\qquad \frac{1+2a}{1/2}, \frac{1}{b}, \frac{2+4a}{1}, \frac{1}{b}, \frac{1+2a}{1/2}$

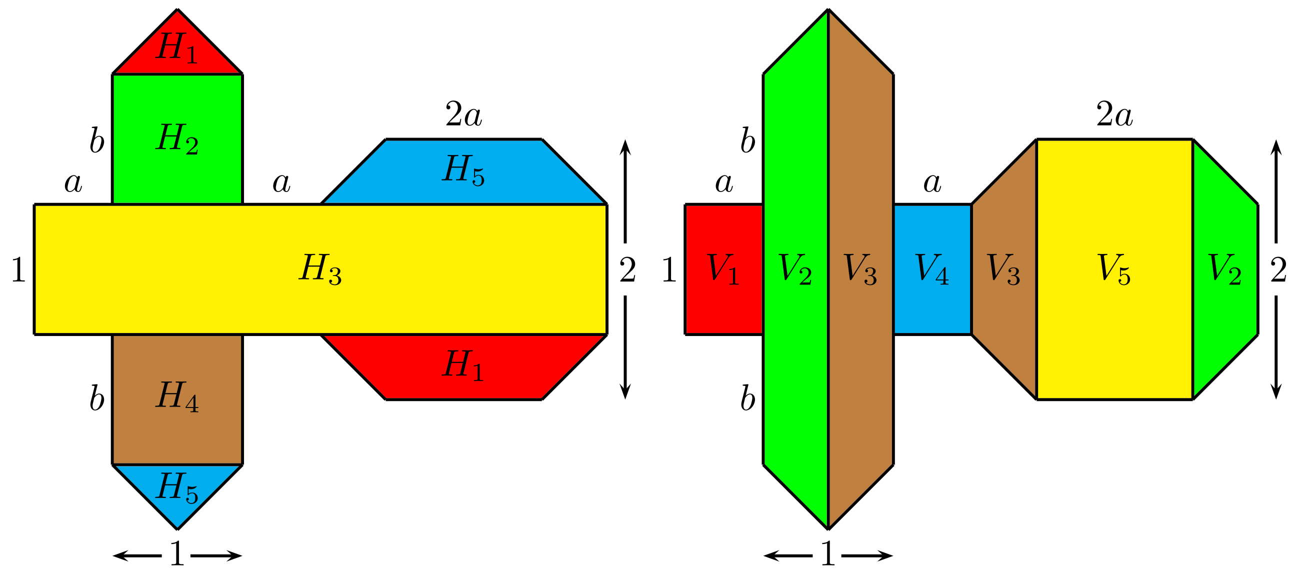
$a = r_1\sqrt{d} + r_2 > 0$, $b = 3r_1\sqrt{d} - 3r_2 - \frac{3}{2} > 0$, $r_1, r_2 \in \mathbb{Q}$, $d \geqslant 2$ squarefree



$\dfrac{\text{length}}{\text{width}}$ of horizontal streets $\qquad \dfrac{1+2a}{1/2}, \dfrac{1}{b}, \dfrac{2+4a}{1}, \dfrac{1}{b}, \dfrac{1+2a}{1/2}$

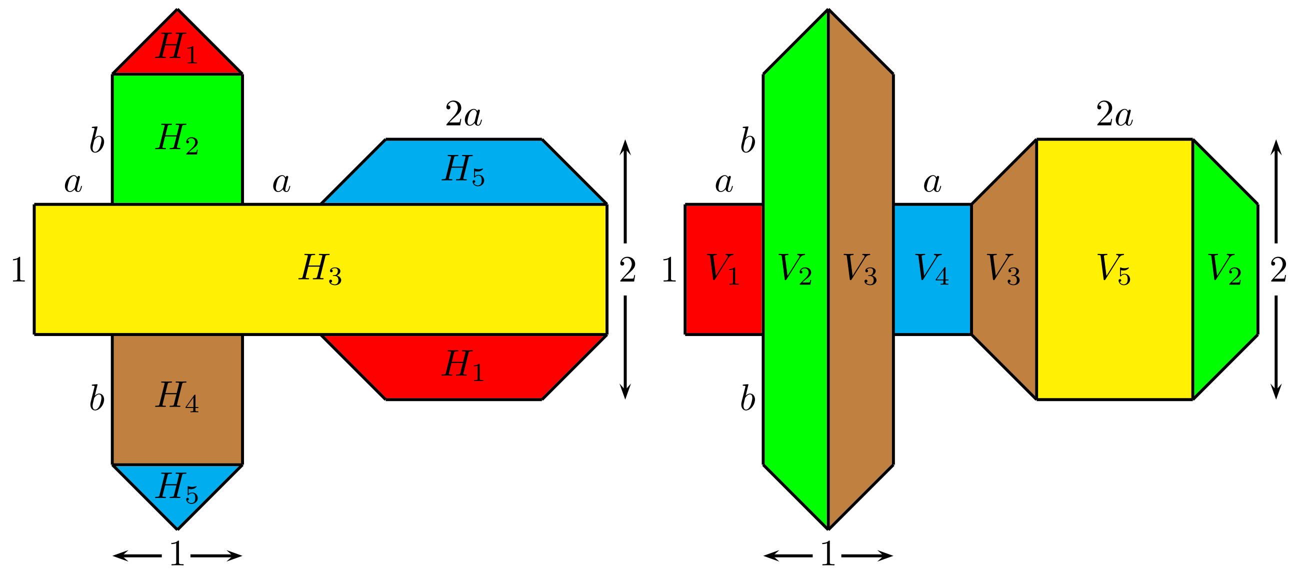
$a = r_1\sqrt{d} + r_2 > 0$, $b = 3r_1\sqrt{d} - 3r_2 - \dfrac{3}{2} > 0$, $r_1, r_2 \in \mathbb{Q}$, $d \geqslant 2$ squarefree

two different shapes with ratio $2b(1+2a) = 3(4r_1^2 d - (2r_2+1)^2) \in \mathbb{Q}$



$\frac{\text{length}}{\text{width}}$ of vertical streets $\quad \frac{1}{a}, \frac{3+2b}{1/2}, \frac{3+2b}{1/2}, \frac{1}{a}, \frac{2}{2a}$

$a = r_1\sqrt{d} + r_2 > 0$, $b = 3r_1\sqrt{d} - 3r_2 - \frac{3}{2} > 0$, $r_1, r_2 \in \mathbb{Q}$, $d \geqslant 2$ squarefree class of cathedral surfaces – McMullen–Mukamel–Wright (2017)

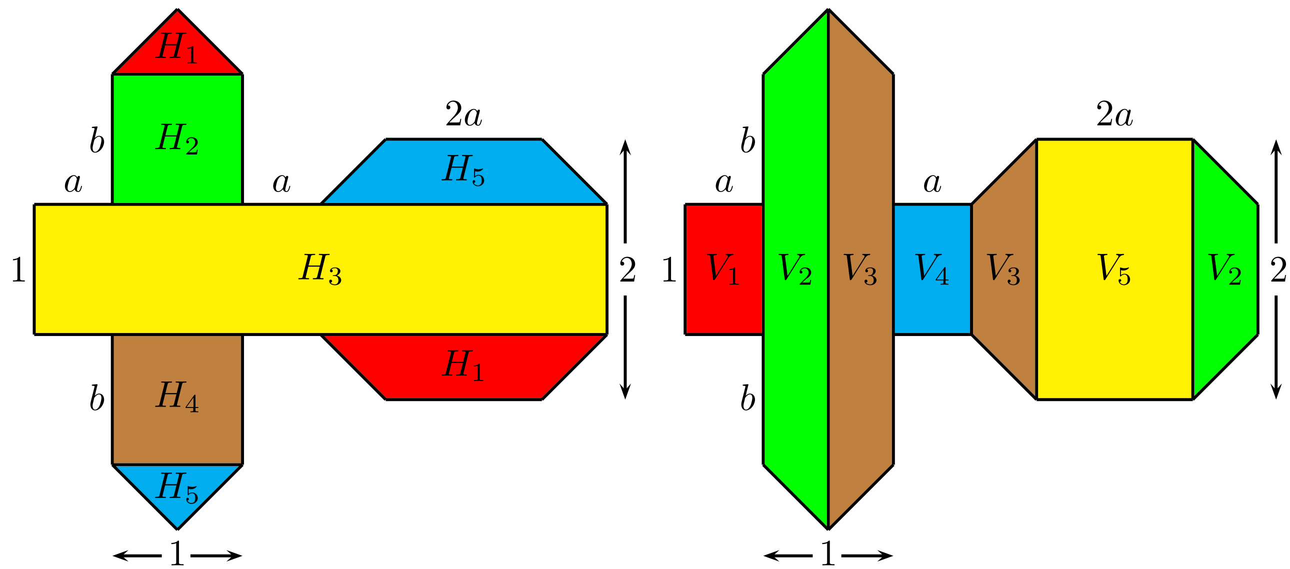


$\frac{\text{length}}{\text{width}}$ of vertical streets $\qquad \frac{1}{a}, \ \frac{3+2b}{1/2}, \ \frac{3+2b}{1/2}, \ \frac{1}{a}, \ \frac{2}{2a}$

$a = r_1\sqrt{d} + r_2 > 0$, $b = 3r_1\sqrt{d} - 3r_2 - \frac{3}{2} > 0$, $r_1, r_2 \in \mathbb{Q}$, $d \geqslant 2$ squarefree

two different shapes with ratio $2a(3+2b) = 12(r_1^2 d - r_2^2) \in \mathbb{Q}$

class of cathedral surfaces – McMullen–Mukamel–Wright (2017)



Beck–C–Yang ($\geqslant$ 2020)

infinitely many explicitly given slopes α

$\mathcal{L}$ – geodesic in cathedral surface with slope α

superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

McMullen (2005, 2006)

classification of all affine-different Veech surfaces of genus 2

– Calta–McMullen L-staircases

– regular decagon surface with parallel edge identification

McMullen (2005, 2006)

classification of all affine-different Veech surfaces of genus 2

– Calta–McMullen L-staircases

– regular decagon surface with parallel edge identification

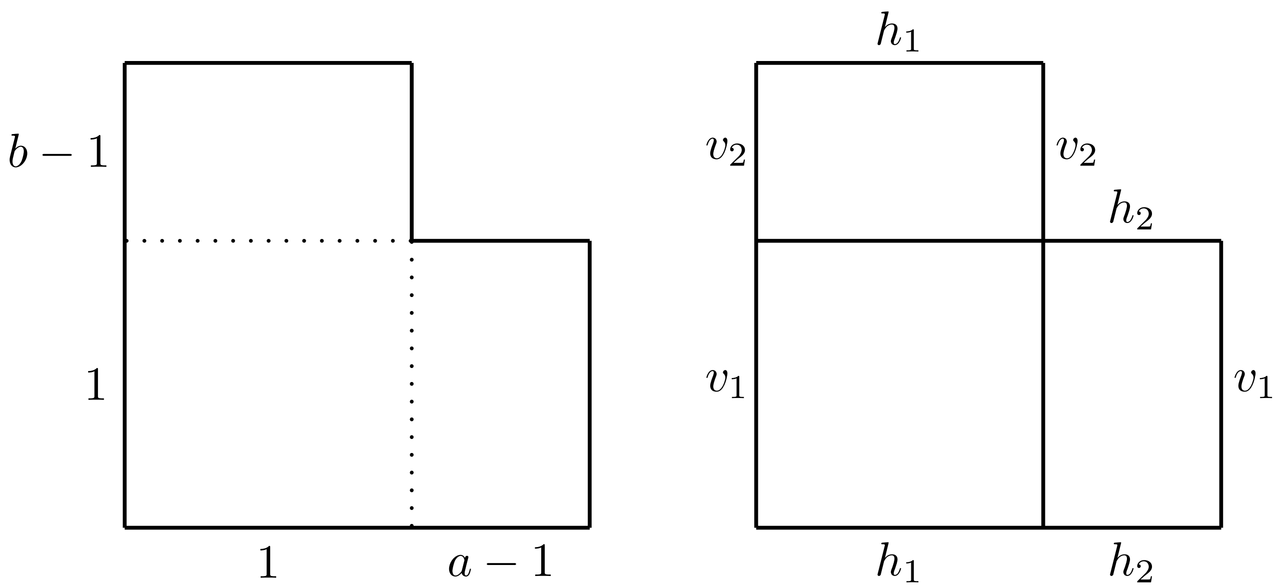
McMullen (2005, 2006)

classification of all affine-different Veech surfaces of genus 2

– Calta–McMullen L-staircases

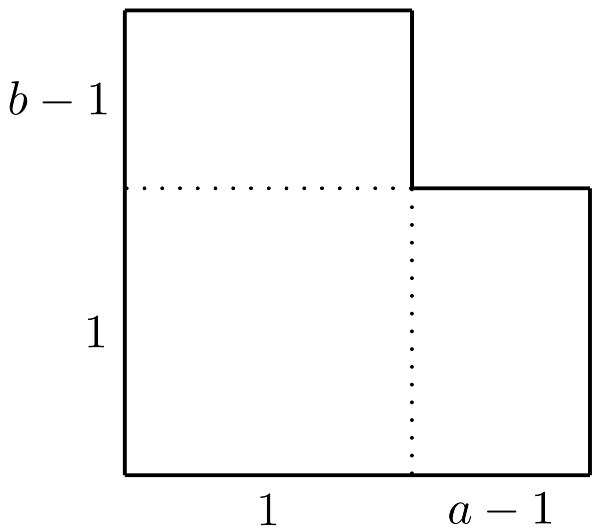
– regular decagon surface with parallel edge identification

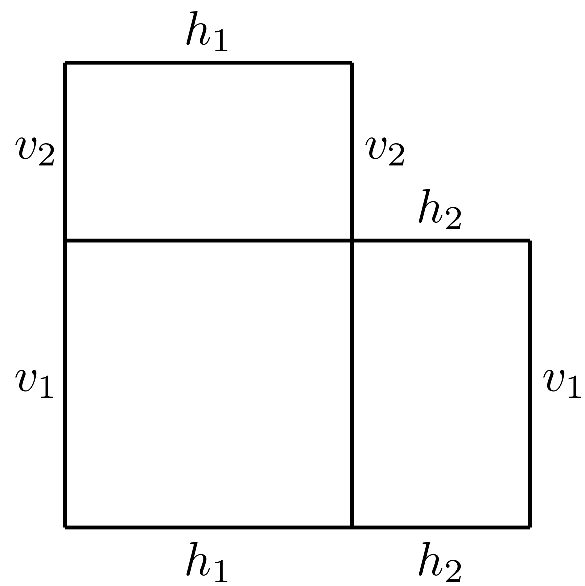
generalization of the L-shape region and L-surface



Calta (2004) $\oplus$ McMullen (2003) – L-staircases

generalization of the L-shape region and L-surface





$a = r_1\sqrt{d} + r_2$, $b = r_1\sqrt{d} + 1 - r_2$, $r_1, r_2 \in \mathbb{Q}$, $d \geqslant 2$ squarefree

street-rational polyrectangle surface

Beck–C–Yang ($\geqslant$ 2020)

infinitely many explicitly given slopes α

$\mathcal{L}$ – geodesic in any surface of C–McM family with slope α

superdensity of $\mathcal{L}$

can compute irregularity exponent

time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets

Beck–C–Yang ($\geqslant$ 2020)

infinitely many explicitly given slopes α

$\mathcal{L}$ – billiard orbit in any region of C–McM family with initial slope α

superdensity of $\mathcal{L}$

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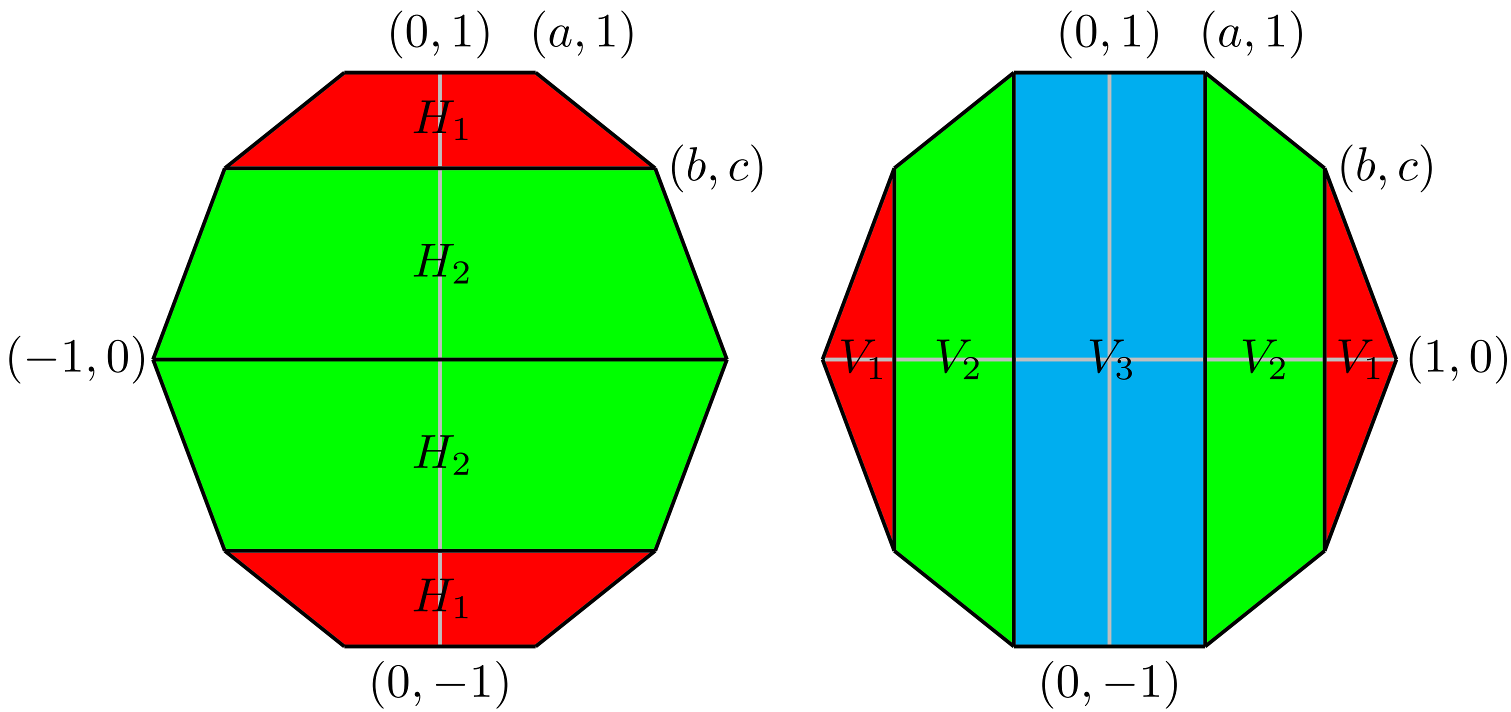
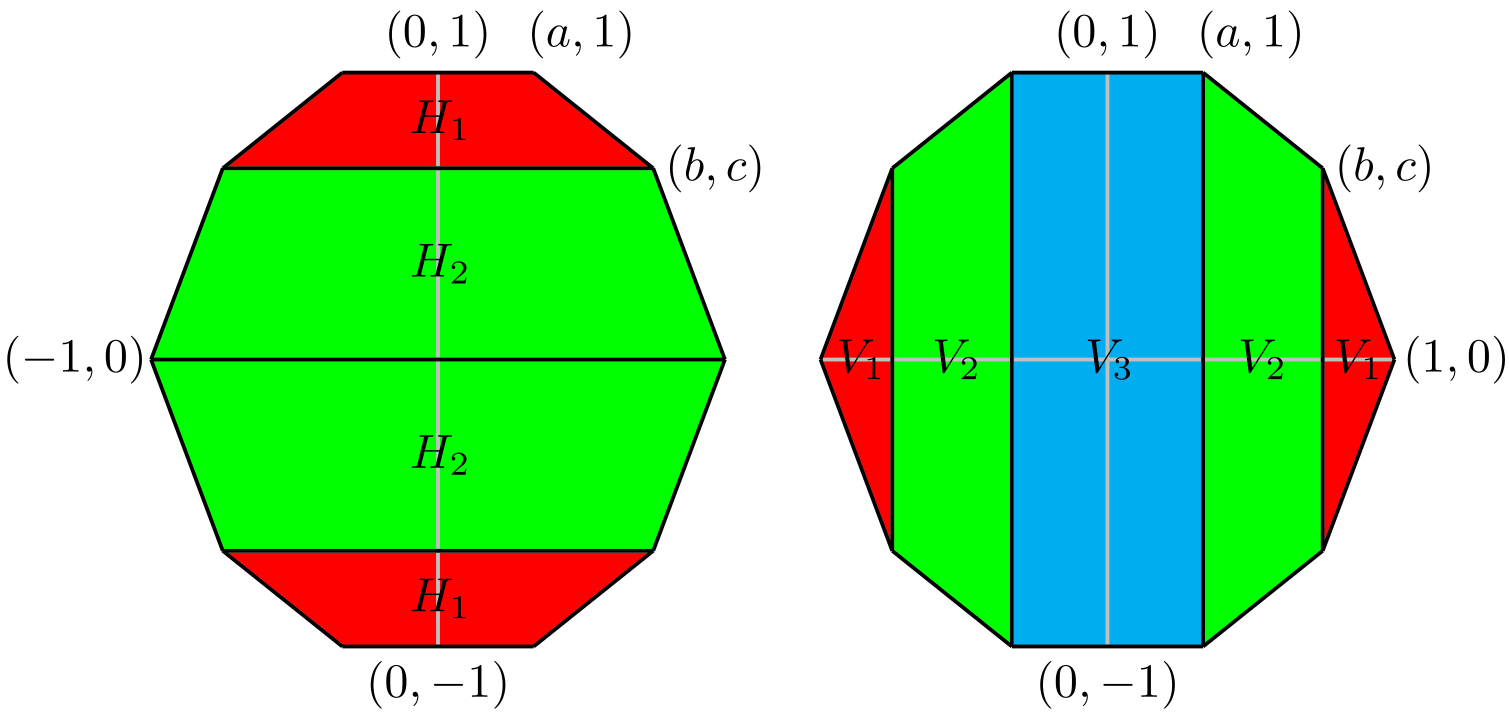
time-quantitative equidistribution of $\mathcal{L}$ relative to all convex sets



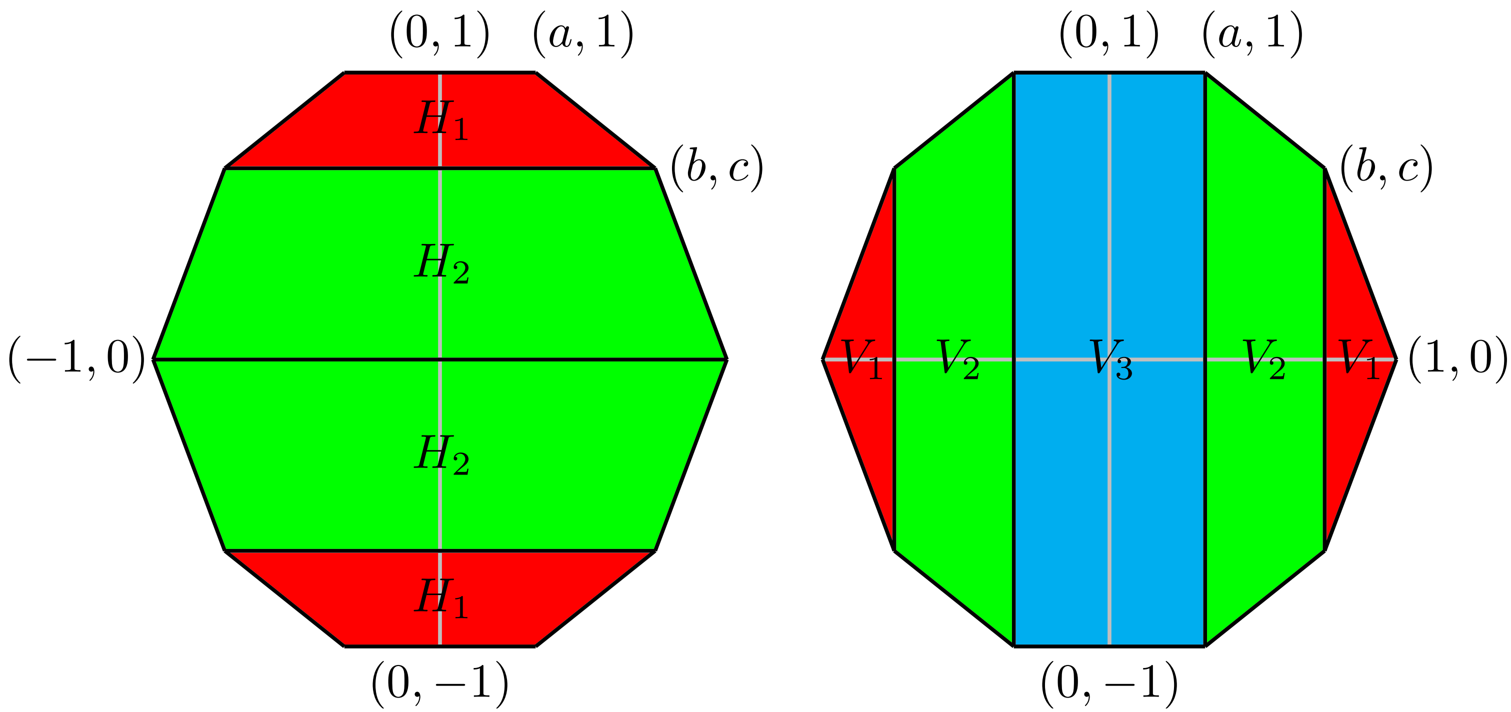
width 2, height 2

a family of genus 2 non-regular decagon surfaces – not Veech



width 2, height 2

a family of genus 2 non-regular decagon surfaces – not Veech



affine-different, width 2, height 2

infinitely many different a, b, c give street-rationality

a family of genus 2 non-regular decagon surfaces S

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McMullen (2005) – translation surface S in genus 2 not Veech

$\Rightarrow$ exist geodesics on S neither dense nor periodic

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$\Rightarrow$ exist geodesics on S dense but not equidistributed

Beck–C–Yang ($\geqslant$ 2020) – shortline method

$\Rightarrow$ exist geodesics on S with superdensity

$\Rightarrow$ exist geodesics on S with time-quantitative equidistribution

THANK YOU