BILLIARD ORBITS AND GEODESICS IN NON-INTEGRABLE FLAT DYNAMICAL SYSTEMS (PART I)

William Chen
Macquarie University Sydney

Workshop on Discrepancy Theory and Applications
Centre International de Rencontres Mathématiques
Luminy
November/December 2020

József Beck
Michael Donders
Yuxuan Yang
C

quintessential integrable dynamical system

linear 1-direction geodesics in flat torus in any dimension $\geqslant 2$

quintessential integrable dynamical system

linear 1-direction geodesics in flat torus in any dimension $\geqslant 2$

Kronecker–Weyl equidistribution theorem

$1, \gamma_1, \ldots, \gamma_m$ linearly independent over $\mathbb{Q}$

$\mathcal{L}$ – infinite half line with direction vector $\mathbf{v} = (1, \gamma_1, \ldots, \gamma_m) \in \mathbb{R}^{m+1}$

$\Rightarrow$ $\mathcal{L}$ uniformly distributed in unit torus $[0, 1]^{m+1}$

quintessential integrable dynamical system

linear 1-direction geodesics in flat torus in any dimension $\geqslant 2$

Kronecker–Weyl equidistribution theorem

$1, \gamma_1, \ldots, \gamma_m$ linearly independent over $\mathbb{Q}$

$\mathcal{L}$ − infinite half line with direction vector $\mathbf{v} = (1, \gamma_1, \ldots, \gamma_m) \in \mathbb{R}^{m+1}$

$\Rightarrow$ $\mathcal{L}$ uniformly distributed in unit torus $[0, 1]^{m+1}$

$\mathbf{v} = (1, \gamma) \in \mathbb{R}^2$

$\gamma \notin \mathbb{Q} \Rightarrow \mathcal{L}$ uniformly distributed in unit torus $[0, 1]^2$

quintessential integrable dynamical system

linear 1-direction geodesics in flat torus in any dimension $\geqslant 2$

Kronecker–Weyl equidistribution theorem

$1, \gamma_1, \ldots, \gamma_m$ linearly independent over $\mathbb{Q}$

$\mathcal{L}$ – infinite half line with direction vector $\mathbf{v} = (1, \gamma_1, \ldots, \gamma_m) \in \mathbb{R}^{m+1}$

$\Rightarrow \mathcal{L}$ uniformly distributed in unit torus $[0,1]^{m+1}$

$\mathbf{v} = (1, \gamma) \in \mathbb{R}^2$

$\gamma \notin \mathbb{Q} \Rightarrow \mathcal{L}$ uniformly distributed in unit torus $[0,1]^2$

$\gamma \in \mathbb{Q} \Rightarrow \mathcal{L}$ periodic in unit torus $[0,1]^2$

quintessential integrable dynamical system

linear 1-direction geodesics in flat torus in any dimension $\geqslant 2$

Kronecker–Weyl equidistribution theorem

$1, \gamma_1, \ldots, \gamma_m$ linearly independent over $\mathbb{Q}$

$\mathcal{L}$ – infinite half line with direction vector $\mathbf{v} = (1, \gamma_1, \ldots, \gamma_m) \in \mathbb{R}^{m+1}$

$\Rightarrow \mathcal{L}$ uniformly distributed in unit torus $[0, 1]^{m+1}$

$\mathbf{v} = (1, \gamma) \in \mathbb{R}^2 \Rightarrow$ uniform-periodic dichotomy

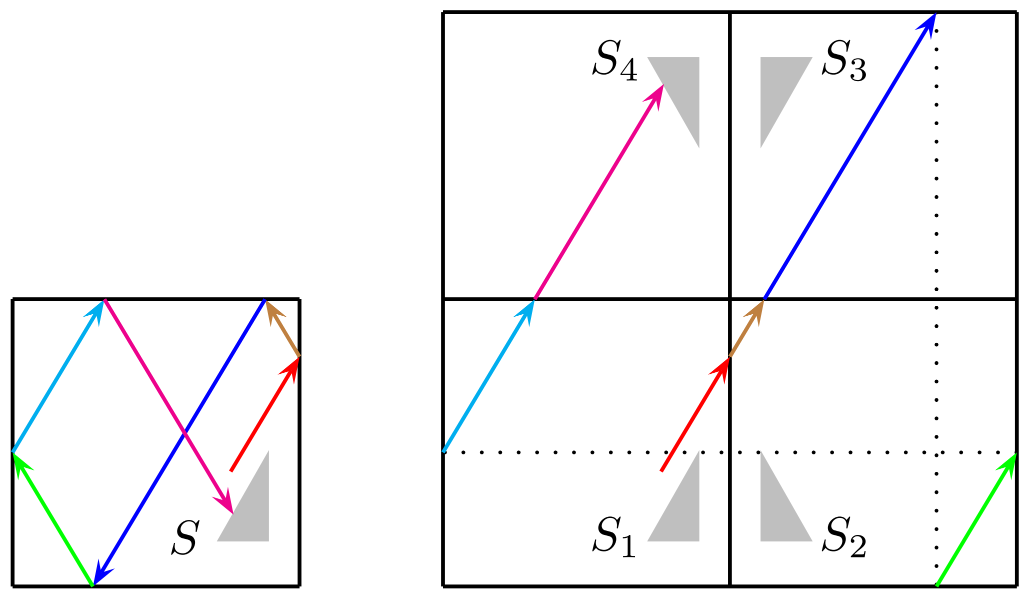
$\gamma \notin \mathbb{Q} \Rightarrow \mathcal{L}$ uniformly distributed in unit torus $[0, 1]^2$

$\gamma \in \mathbb{Q} \Rightarrow \mathcal{L}$ periodic in unit torus $[0, 1]^2$

König–Szücs (1913) – unfolding

billiard orbit in unit square $[0,1]^2$

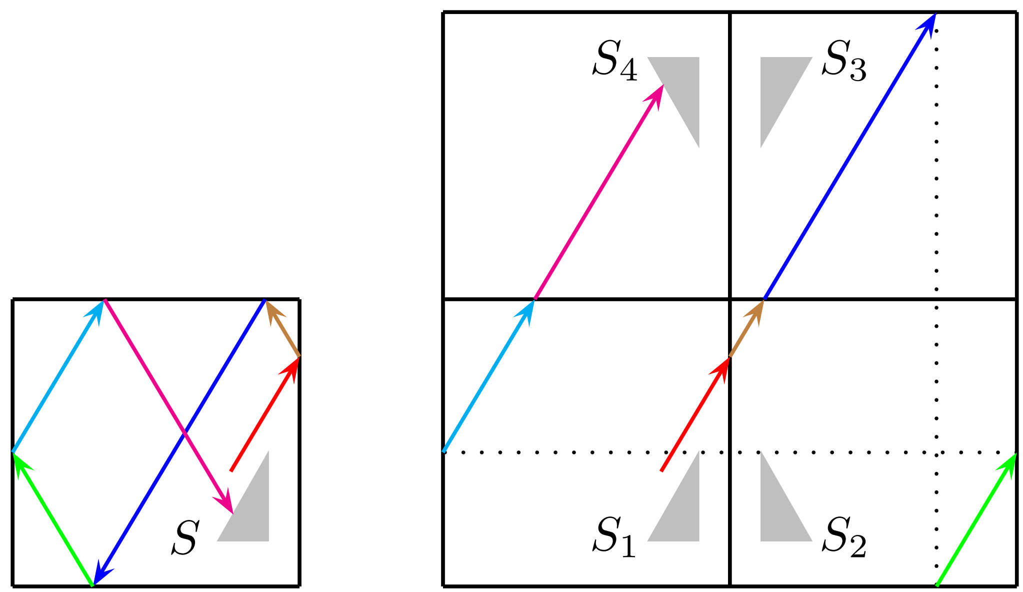
$\hookrightarrow$ 1-direction geodesic on torus $[0,2]^2$



König–Szücs (1913) – unfolding

billiard orbit in unit square $[0,1]^2$

$\hookrightarrow$ 1-direction geodesic on torus $[0,2]^2$



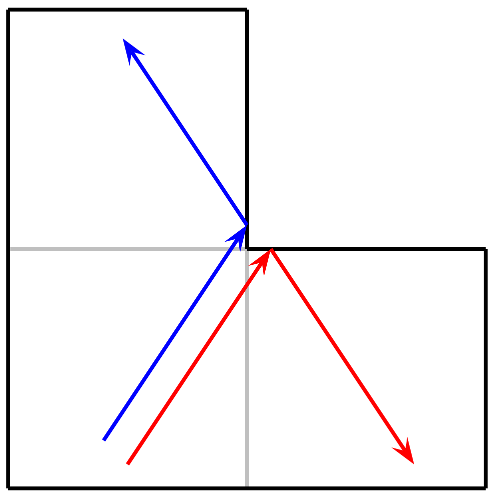
reduce study of billiard orbits to study of 1-direction geodesics

two particles moving on parallel trajectories and initially close together

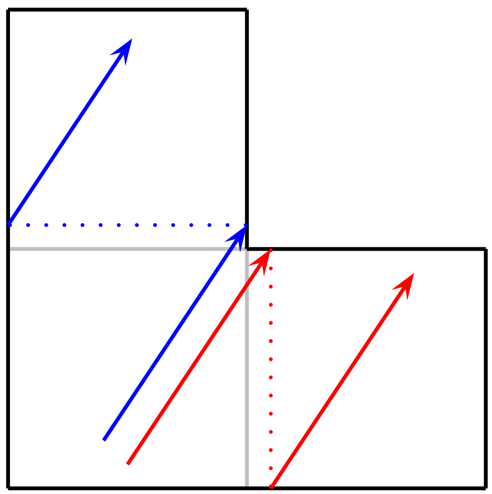
remain close forever

non-integrable systems – split near singular points

billiard on L-shape region

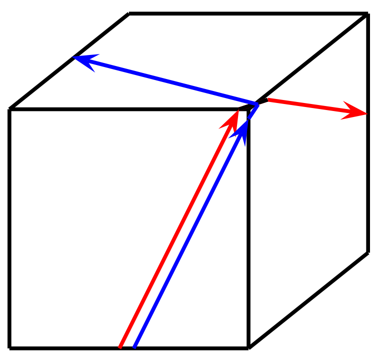


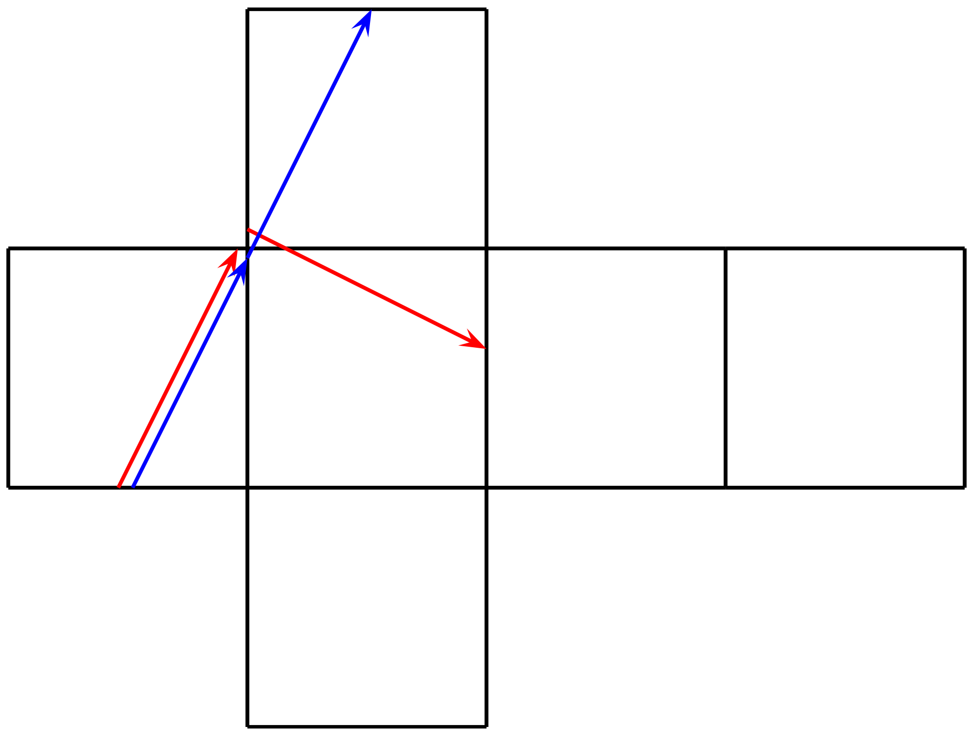
1-direction geodesics on L-surface



non-integrable systems – split near singular points

geodesics on cube surface





$\mathcal{F}$ – flat surface, every face a polygon with angles in $\mathbb{Q}\pi$

$\mathcal{F}$ – flat surface, every face a polygon with angles in $\mathbb{Q}\pi$

Katok–Zemlyakov (1975)

$\mathcal{L}$ – geodesic on $\mathcal{F}$ with non-pathological slope and initial point

$\Rightarrow \mathcal{L}$ dense on $\mathcal{F}$

$\mathcal{F}$ – flat surface, every face a polygon with angles in $\mathbb{Q}\pi$

Katok–Zemlyakov (1975)

$\mathcal{L}$ – geodesic on $\mathcal{F}$ with non-pathological slope and initial point

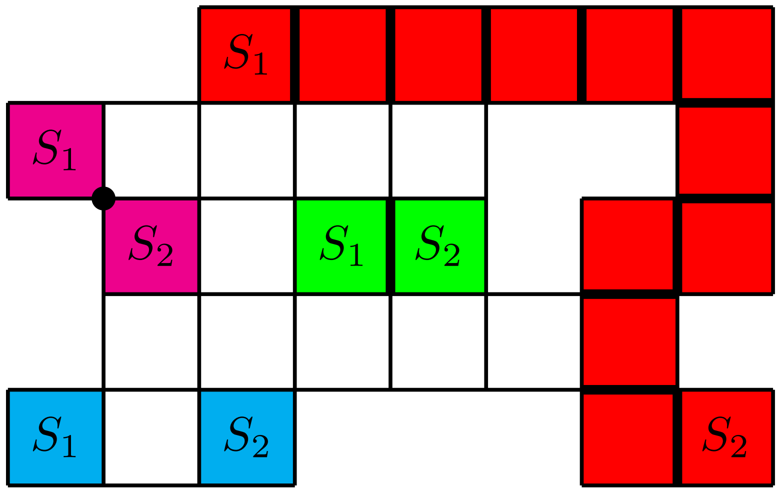
$\Rightarrow$ $\mathcal{L}$ dense on $\mathcal{F}$

Kerckhoff–Masur–Smillie (1986)

$\mathcal{L}$ – geodesic on $\mathcal{F}$ with almost any slope and initial point

$\Rightarrow$ $\mathcal{L}$ equidistributed in $\mathcal{F}$

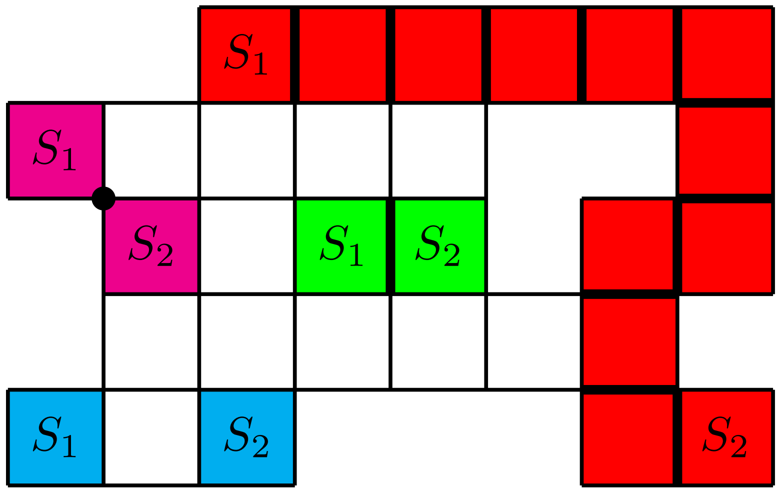
finite polysquare region – region with finitely many unit size squares



disjoint or common vertex or common edge

chain with common edges

finite polysquare region – region with finitely many unit size squares



disjoint or common vertex or common edge

chain with common edges

edge identification $\hookrightarrow$ finite polysquare surface

Gutkin (1984) $\oplus$ Veech (1987)

$\mathcal{L}$ − billiard in finite polysquare region, irrational initial slope

$\mathcal{L}$ − 1-direction geodesic on finite polysquare surface, irrational slope

$\Rightarrow$ $\mathcal{L}$ equidistributed unless it hits a vertex

uniform-periodic dichotomy

Gutkin (1984) $\oplus$ Veech (1987)

$\mathcal{L}$ – billiard in finite polysquare region, irrational initial slope

$\mathcal{L}$ – 1-direction geodesic on finite polysquare surface, irrational slope

$\Rightarrow$ $\mathcal{L}$ equidistributed unless it hits a vertex

uniform-periodic dichotomy

Veech (1992)

$\mathcal{L}$ – billiard in regular polygon

$\Rightarrow$ uniform-periodic dichotomy

qualitative results concerning non-integrable flat systems

no information on necessary time range in any of these results

proof uses Birkhoff's ergodic theorem

time-qualitative – requires unlimited time range

qualitative results concerning non-integrable flat systems

no information on necessary time range in any of these results

proof uses Birkhoff's ergodic theorem

time-qualitative – requires unlimited time range

want time-quantitative results on density and uniformity

qualitative results concerning non-integrable flat systems

no information on necessary time range in any of these results

proof uses Birkhoff's ergodic theorem

time-qualitative – requires unlimited time range

want time-quantitative results on density and uniformity

non-ergodic approach – combinatorics, number theory, linear algebra

shortline method

qualitative results concerning non-integrable flat systems

no information on necessary time range in any of these results

proof uses Birkhoff's ergodic theorem

time-qualitative – requires unlimited time range

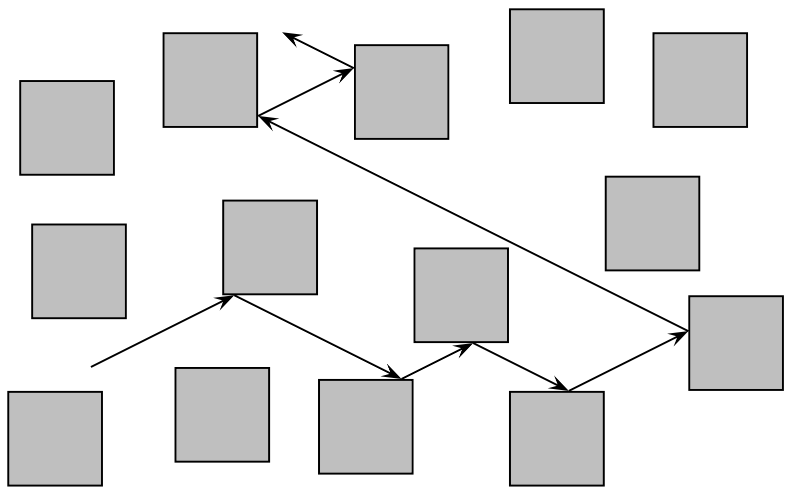
want time-quantitative results on density and uniformity

non-ergodic approach – combinatorics, number theory, linear algebra

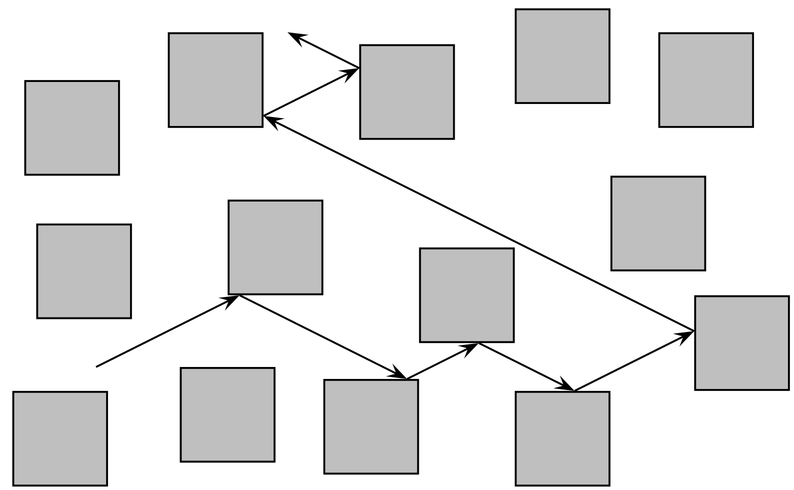
shortline method

eigenvalue-free version – size magnification of intervals

eigenvalue-based version – eigenvalues/vectors of transition matrices



Ehrenfest–Ehrenfest (1912) – (irregular) wind-tree model



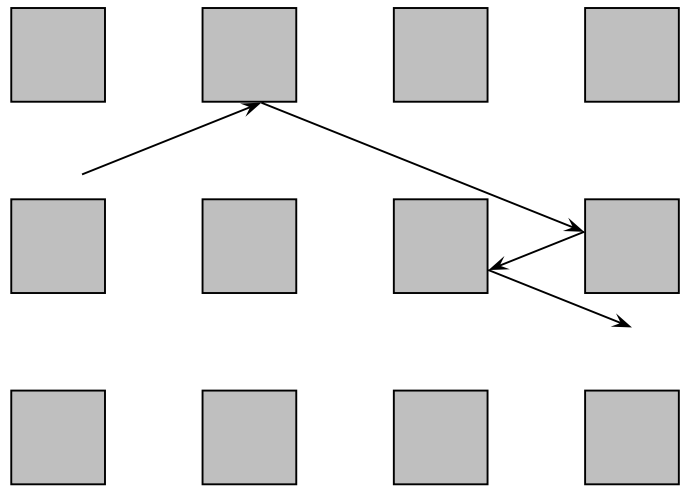
Sabogal–Troubetskoy (2016)

$\mathcal{W}$ – generic configuration (in the sense of Baire) of the model

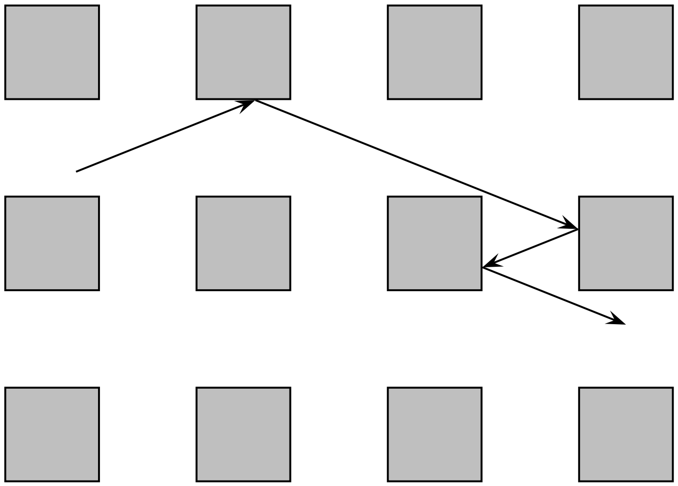
$\Rightarrow$ billiard in $\mathcal{W}$ is ergodic in almost every initial direction

no explicit configuration given

Hardy–Weber (1980) – periodic wind-tree model



Hardy–Weber (1980) – periodic wind-tree model



Fraczek–Ulcigrai (2014)

$\mathcal{W}$ – periodic configuration e.g. side length 1 and gap 1

$\Rightarrow$ billiard in $\mathcal{W}$ for almost every initial direction is not dense

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{W}$ – periodic configuration – side length 1 and gap 1 $\Rightarrow$

- there exist $\aleph_0$-many explicitly given initial directions α such that there are billiard orbits dense in $\mathcal{W}$ with initial direction α
- density established in time-quantitative sense

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{W}$ – periodic configuration – side length 1 and gap 1 $\Rightarrow$

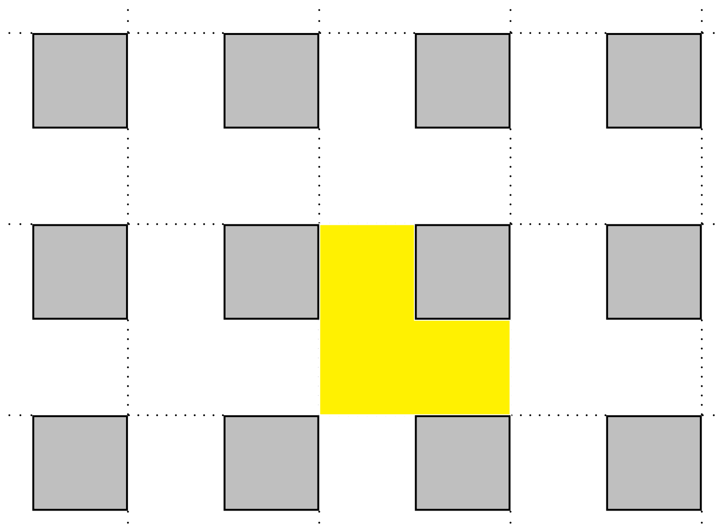
$\circ$ there exist $\aleph_0$-many explicitly given initial directions α such that

there are billiard orbits dense in $\mathcal{W}$ with initial direction α

$\circ$ density established in time-quantitative sense

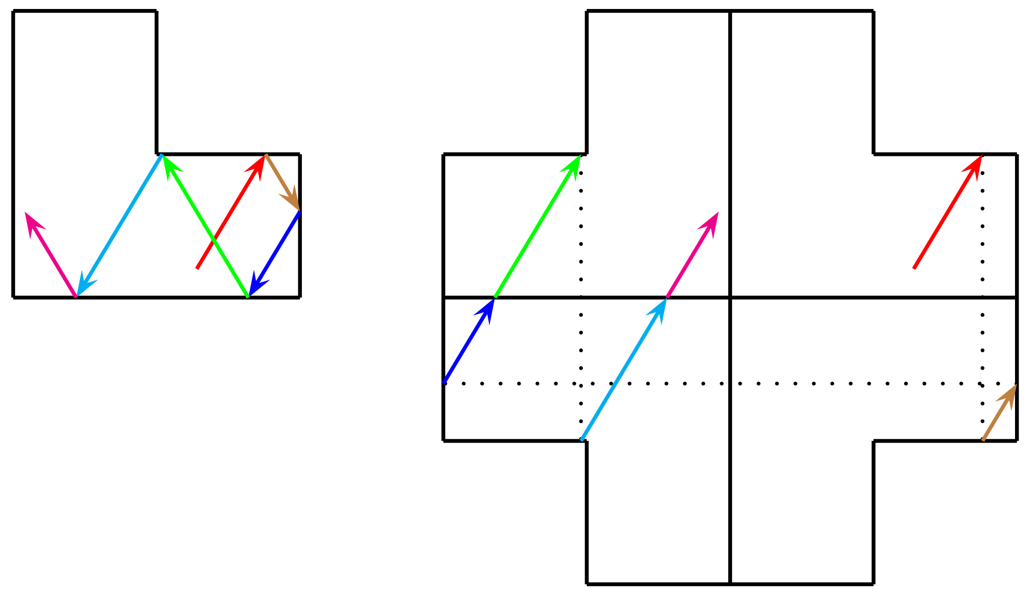
similar conclusion for c-many aperiodic modifications of $\mathcal{W}$

$\mathcal{W}$ – periodic configuration – side length 1 and gap 1



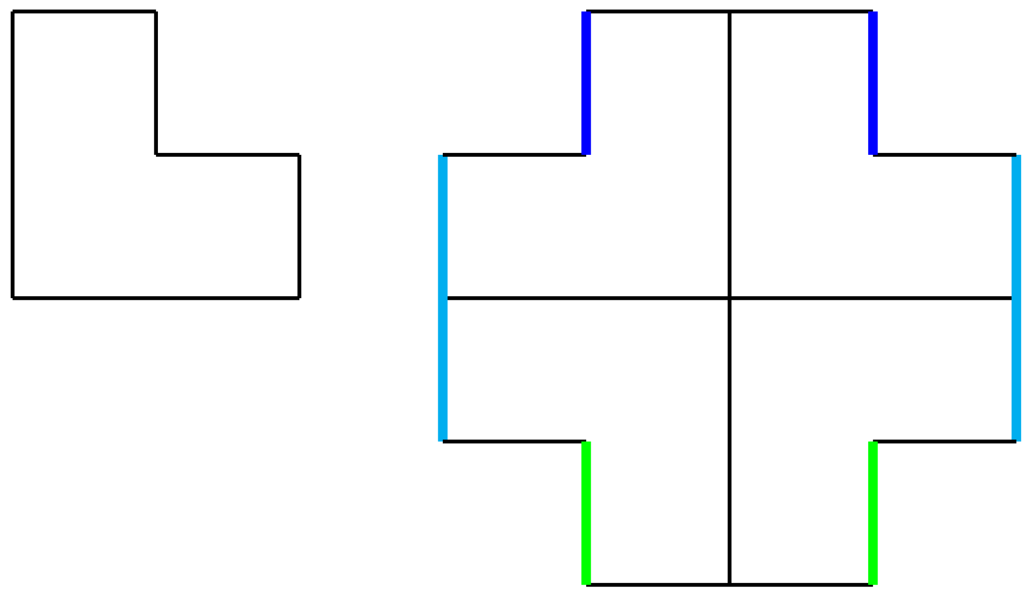
study billiard in L-shape region first

unfolding a la König–Szücs



$\hookrightarrow$ 1-direction geodesics in L-cross surface billiard in L-shape region

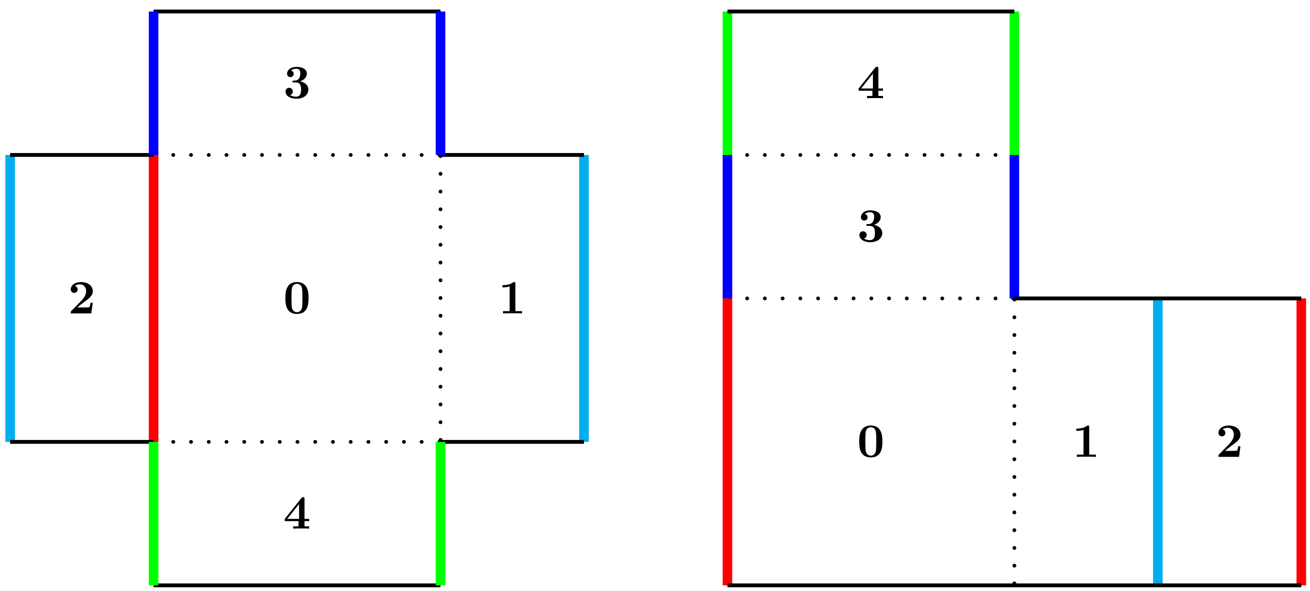
$\hookrightarrow$ 1-direction geodesic on L-cross surface



vertical edge identification

horizontal edge identification not illustrated billiard in L-shape region

$\hookrightarrow$ 1-direction geodesic on L-cross surface



$\hookrightarrow$ 1-direction geodesic on L-surface

V – 1-direction half-infinite geodesic on L-surface $\mathcal{P}$

$V(\ell)$ – initial segment of V of length ℓ

V superdense on $\mathcal{P}$ if there exists constant C such that for any $n \in \mathbb{N}$

$V(Cn)$ is $1/n$-close to every point $Q \in \mathcal{P}$

V – 1-direction half-infinite geodesic on L-surface $\mathcal{P}$

$V(\ell)$ – initial segment of V of length ℓ

V superdense on $\mathcal{P}$ if there exists constant C such that for any $n \in \mathbb{N}$

$V(Cn)$ is $1/n$-close to every point $Q \in \mathcal{P}$

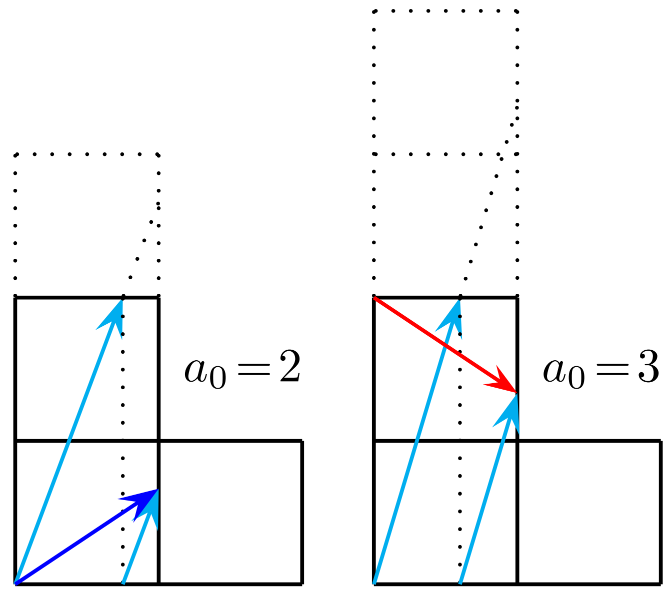
Beck–Donders–Yang (2020) $\oplus$ Beck–C–Yang ($\geqslant$ 2020)

$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$ badly approximable

where $a_0, a_1, a_2, a_3, \ldots \in \mathbb{N}$ are even and bounded

$\Rightarrow$ any V with slope α is superdense on L-surface $\mathcal{P}$

first almost vertical detour crossing has slope α

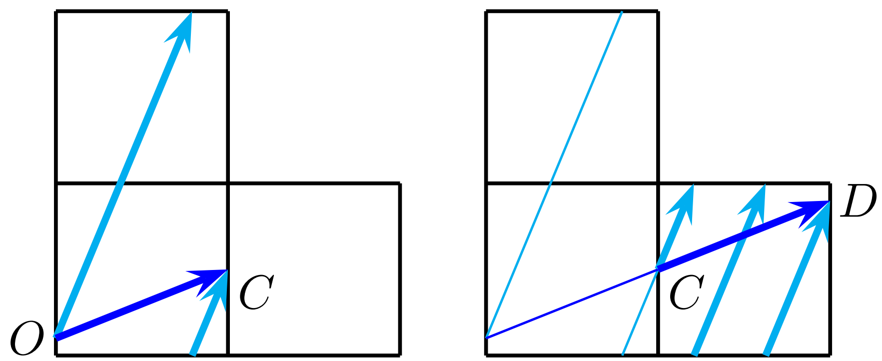


$a_0 = 2 \hookrightarrow$ shortcut has positive slope $\alpha - a_0 = \alpha_1^{-1}$

$\alpha_1 = [a_1; a_2, a_3, \ldots]$ obtained from $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ by shift

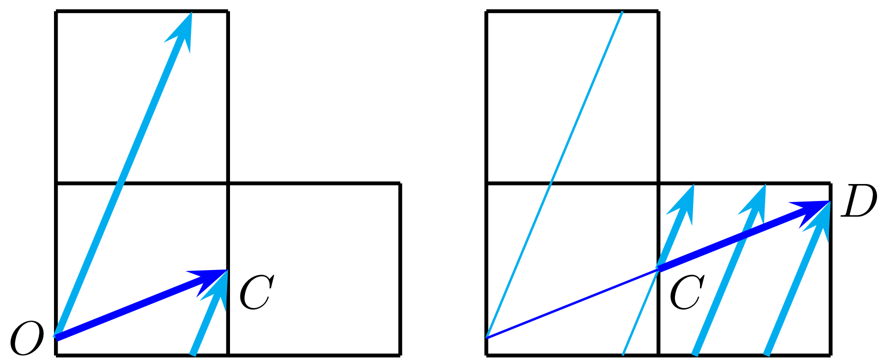
$a_0 = 3 \hookrightarrow$ shortcut has negative slope – want to avoid this

first almost vertical detour crossing



first almost horizontal shortcut – intersect at O and C

first almost vertical detour crossing



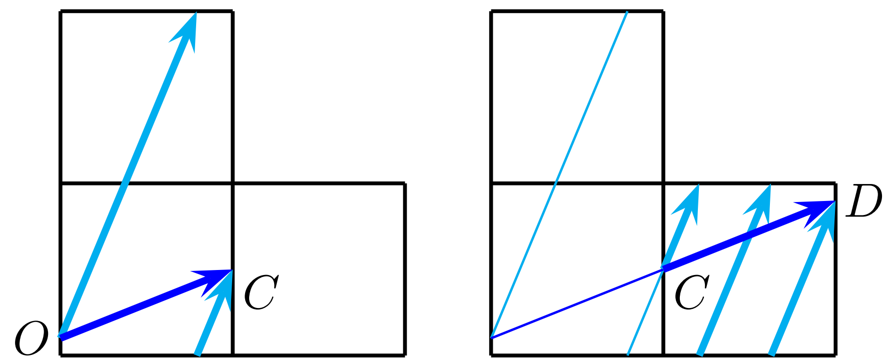
first almost horizontal shortcut – intersect at O and C

second almost vertical detour crossing

second almost horizontal shortcut – intersect at C and D

and so on

first almost vertical detour crossing



first almost horizontal shortcut – intersect at O and C

second almost vertical detour crossing

second almost horizontal shortcut – intersect at C and D

and so on

geodesic V and shortline H_1 – vertical same edge cutting property

almost vertical geodesic V and almost horizontal shortline H_1

– vertical same edge cutting property

almost vertical geodesic V and almost horizontal shortline H_1

– vertical same edge cutting property

almost horizontal geodesic H_1 and almost vertical shortline V_2

– horizontal same edge cutting property

almost vertical geodesic V and almost horizontal shortline H_1

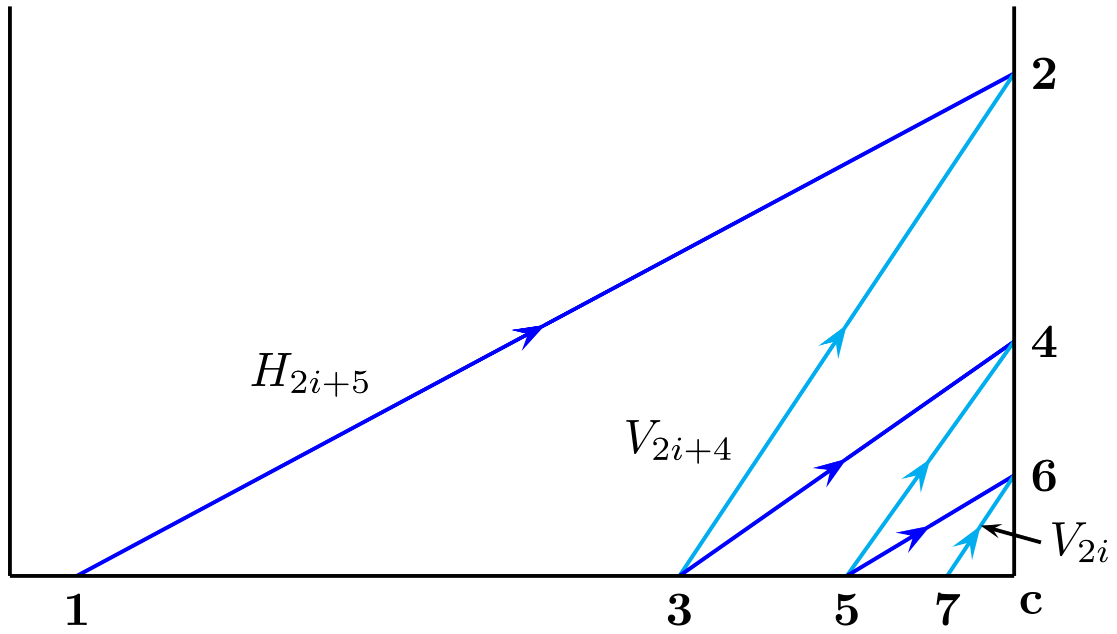
– vertical same edge cutting property

almost horizontal geodesic H_1 and almost vertical shortline V_2

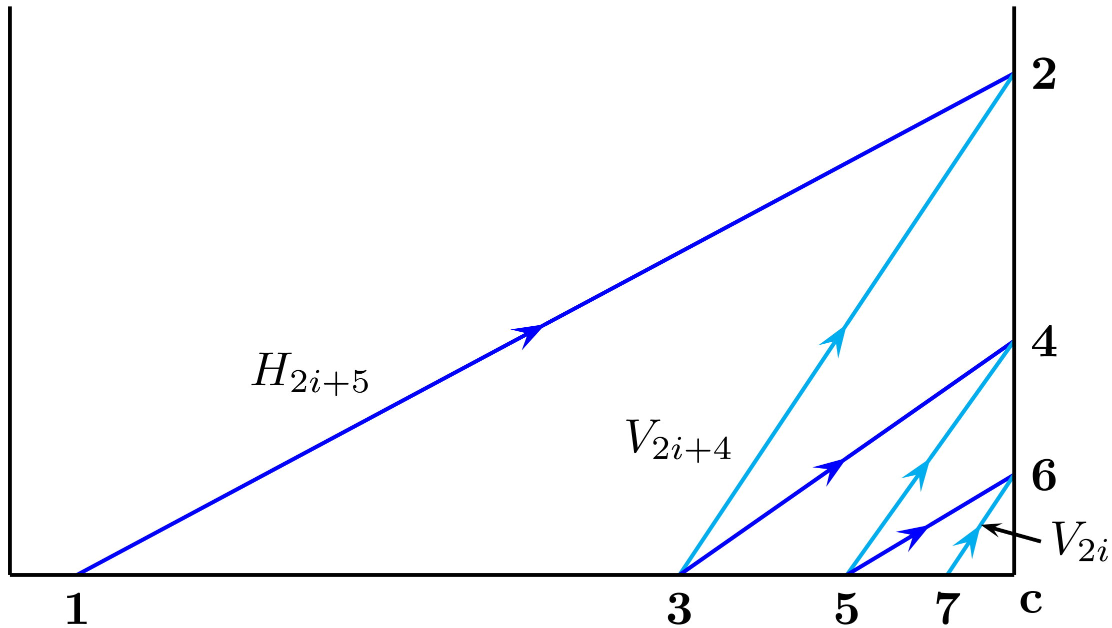
– horizontal same edge cutting property

$$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \ldots$$

$$V_{2i} \xrightarrow{\text{sv}} H_{2i+1} \xrightarrow{\text{sh}} V_{2i+2} \xrightarrow{\text{sv}} H_{2i+3} \xrightarrow{\text{sh}} V_{2i+4} \xrightarrow{\text{sv}} H_{2i+5} \xrightarrow{\text{sh}} \ldots$$



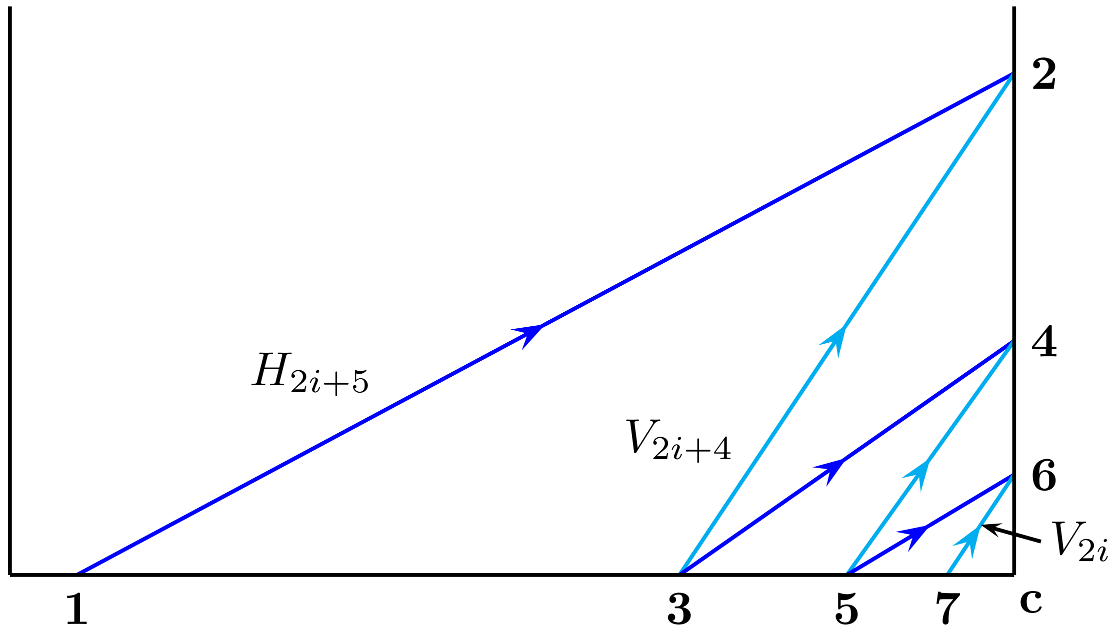
$$V_{2i} \xrightarrow{\text{sv}} H_{2i+1} \xrightarrow{\text{sh}} V_{2i+2} \xrightarrow{\text{sv}} H_{2i+3} \xrightarrow{\text{sh}} V_{2i+4} \xrightarrow{\text{sv}} H_{2i+5} \xrightarrow{\text{sh}} \dots$$



$$\frac{\text{length}(\mathbf{1c})}{\text{length}(\mathbf{2c})} = \alpha_{2i+5}, \quad \frac{\text{length}(\mathbf{2c})}{\text{length}(\mathbf{3c})} = \alpha_{2i+4}, \quad \frac{\text{length}(\mathbf{3c})}{\text{length}(\mathbf{4c})} = \alpha_{2i+3},$$

$$\frac{\text{length}(\mathbf{4c})}{\text{length}(\mathbf{5c})} = \alpha_{2i+2}, \quad \frac{\text{length}(\mathbf{5c})}{\text{length}(\mathbf{6c})} = \alpha_{2i+1}, \quad \frac{\text{length}(\mathbf{6c})}{\text{length}(\mathbf{7c})} = \alpha_{2i}$$

$$V_{2i} \xrightarrow{\text{sv}} H_{2i+1} \xrightarrow{\text{sh}} V_{2i+2} \xrightarrow{\text{sv}} H_{2i+3} \xrightarrow{\text{sh}} V_{2i+4} \xrightarrow{\text{sv}} H_{2i+5} \xrightarrow{\text{sh}} \ldots$$



$$\frac{\text{length}(\mathbf{1c})}{\text{length}(\mathbf{2c})} = \alpha_{2i+5}, \quad \frac{\text{length}(\mathbf{2c})}{\text{length}(\mathbf{3c})} = \alpha_{2i+4}, \quad \frac{\text{length}(\mathbf{3c})}{\text{length}(\mathbf{4c})} = \alpha_{2i+3},$$

$$\frac{\text{length}(\mathbf{4c})}{\text{length}(\mathbf{5c})} = \alpha_{2i+2}, \quad \frac{\text{length}(\mathbf{5c})}{\text{length}(\mathbf{6c})} = \alpha_{2i+1}, \quad \frac{\text{length}(\mathbf{6c})}{\text{length}(\mathbf{7c})} = \alpha_{2i}$$

exponentially fast zigzagging to a street corner

$$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \ldots$$

V^* – long finite initial segment of V

finitely many whole detour crossings and one fractional detour crossing

$$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \ldots$$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \dots$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

H_1^* – finite segment of H_1

finitely many shortcuts and one fractional shortcut of V^*

$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \ldots$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

H_1^* – finite segment of H_1

finitely many shortcuts and one fractional shortcut of V^*
finitely many whole detour crossings and one fractional detour crossing

$$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \dots$$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

H_1^* – finite segment of H_1 length$(H_1^*) = m_1(1+\alpha_1^2)^{1/2}$

finitely many shortcuts and one fractional shortcut of V^*
finitely many whole detour crossings and one fractional detour crossing

$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \dots$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

H_1^* – finite segment of H_1 length$(H_1^*) = m_1(1+\alpha_1^2)^{1/2}$

finitely many shortcuts and one fractional shortcut of V^*
finitely many whole detour crossings and one fractional detour crossing

V_2^* – finite segment of V_2

finitely many shortcuts and one fractional shortcut of H_1^*

$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \dots$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

H_1^* – finite segment of H_1 length$(H_1^*) = m_1(1+\alpha_1^2)^{1/2}$

finitely many shortcuts and one fractional shortcut of V^*
finitely many whole detour crossings and one fractional detour crossing

V_2^* – finite segment of V_2

finitely many shortcuts and one fractional shortcut of H_1^*
finitely many whole detour crossings and one fractional detour crossing

$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \ldots$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

H_1^* – finite segment of H_1 length$(H_1^*) = m_1(1+\alpha_1^2)^{1/2}$

finitely many shortcuts and one fractional shortcut of V^*
finitely many whole detour crossings and one fractional detour crossing

V_2^* – finite segment of V_2 length$(V_2^*) = m_2(1+\alpha_2^2)^{1/2}$

finitely many shortcuts and one fractional shortcut of H_1^*
finitely many whole detour crossings and one fractional detour crossing

$V \xrightarrow{\text{sv}} H_1 \xrightarrow{\text{sh}} V_2 \xrightarrow{\text{sv}} H_3 \xrightarrow{\text{sh}} V_4 \xrightarrow{\text{sv}} H_5 \xrightarrow{\text{sh}} \dots$

V^* – long finite initial segment of V length$(V^*) = m_0(1+\alpha^2)^{1/2}$

finitely many whole detour crossings and one fractional detour crossing

H_1^* – finite segment of H_1 length$(H_1^*) = m_1(1+\alpha_1^2)^{1/2}$

finitely many shortcuts and one fractional shortcut of V^*
finitely many whole detour crossings and one fractional detour crossing

V_2^* – finite segment of V_2 length$(V_2^*) = m_2(1+\alpha_2^2)^{1/2}$

finitely many shortcuts and one fractional shortcut of H_1^*
finitely many whole detour crossings and one fractional detour crossing

$V^* \xrightarrow{\text{sv}} H_1^* \xrightarrow{\text{sh}} V_2^* \xrightarrow{\text{sv}} H_3^* \xrightarrow{\text{sh}} V_4^* \xrightarrow{\text{sv}} H_5^* \xrightarrow{\text{sh}} \dots$

$$V^* \xrightarrow{\text{sv}} H_1^* \xrightarrow{\text{sh}} V_2^* \xrightarrow{\text{sv}} H_3^* \xrightarrow{\text{sh}} V_4^* \xrightarrow{\text{sv}} H_5^* \xrightarrow{\text{sh}} \ldots$$

$$\text{length}(V^*) = m_0(1+\alpha^2)^{1/2}$$

$$\text{length}(H_1^*) = m_1(1+\alpha_1^2)^{1/2}$$

$$\text{length}(V_2^*) = m_2(1+\alpha_2^2)^{1/2}$$

$$V^* \xrightarrow{\text{sv}} H_1^* \xrightarrow{\text{sh}} V_2^* \xrightarrow{\text{sv}} H_3^* \xrightarrow{\text{sh}} V_4^* \xrightarrow{\text{sv}} H_5^* \xrightarrow{\text{sh}} \dots$$

$$\text{length}(V^*) = m_0(1+\alpha^2)^{1/2}$$

$$\text{length}(H_1^*) = m_1(1+\alpha_1^2)^{1/2} \qquad \alpha_1 m_1 = m_0$$

$$\text{length}(V_2^*) = m_2(1+\alpha_2^2)^{1/2} \qquad \alpha_2 m_2 = m_1$$

$$m_0 = m_\ell \alpha_1 \dots \alpha_\ell$$

$$V^* \xrightarrow{\text{sv}} H_1^* \xrightarrow{\text{sh}} V_2^* \xrightarrow{\text{sv}} H_3^* \xrightarrow{\text{sh}} V_4^* \xrightarrow{\text{sv}} H_5^* \xrightarrow{\text{sh}} \dots$$

$\text{length}(V^*) = m_0(1 + \alpha^2)^{1/2}$

$\text{length}(H_1^*) = m_1(1 + \alpha_1^2)^{1/2}$ $\qquad \alpha_1 m_1 = m_0$

$\text{length}(V_2^*) = m_2(1 + \alpha_2^2)^{1/2}$ $\qquad \alpha_2 m_2 = m_1$

$m_0 = m_\ell \alpha_1 \dots \alpha_\ell$

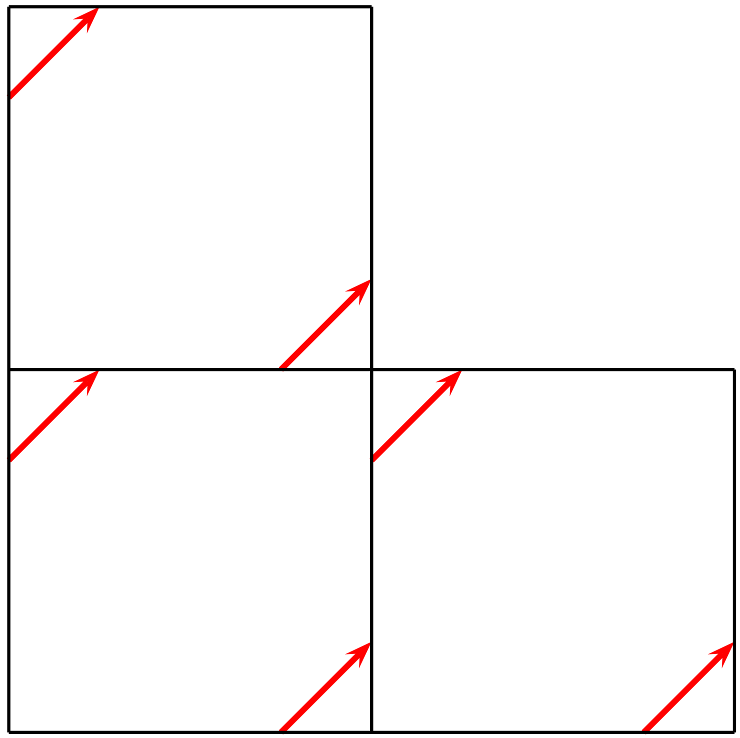
assumption – α is badly approximable

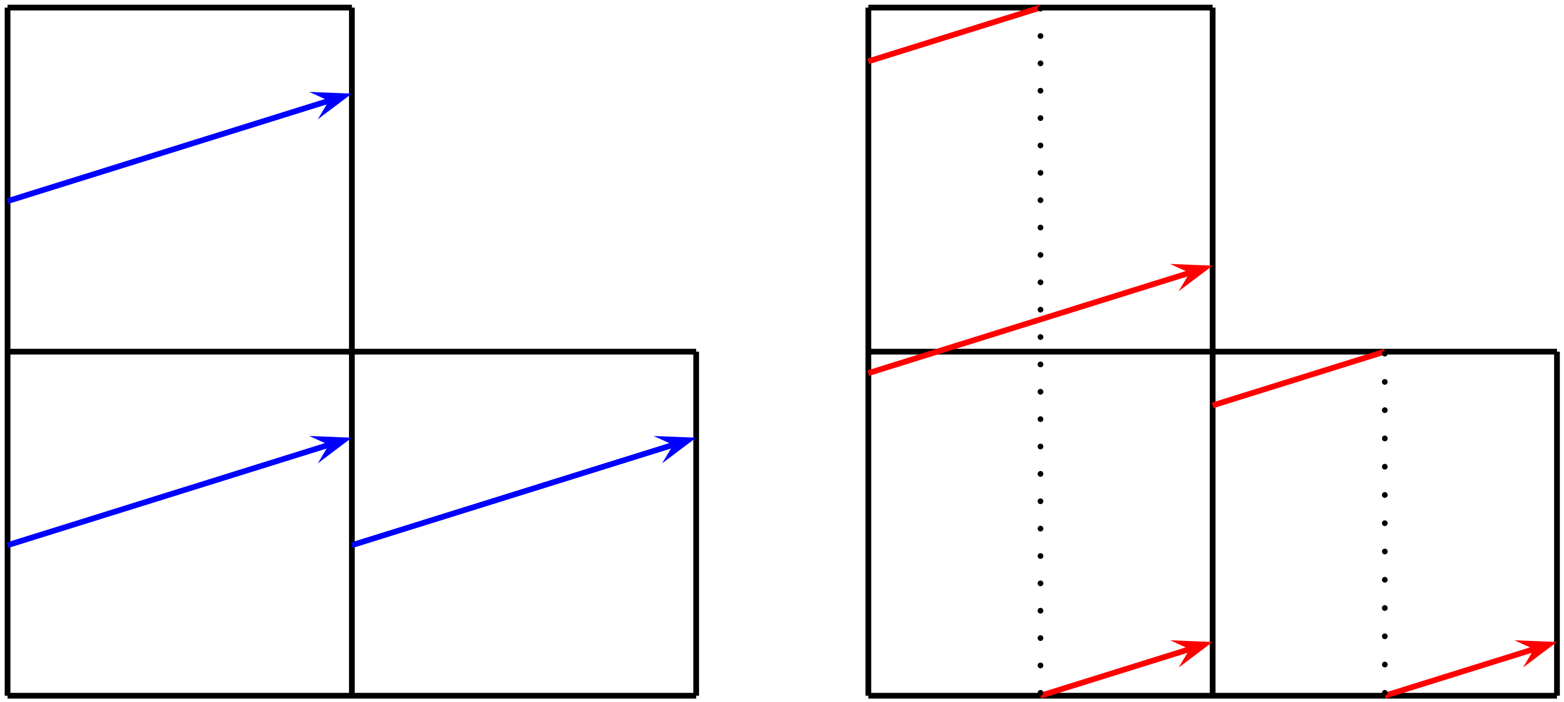
$a_0, a_1, a_2, a_3, \dots < U$ for some integer U

$\alpha, \alpha_1, \alpha_2, \alpha_3, \dots < U$

claim − there is some m_ℓ satisfying $2U + 1 \leqslant m_\ell \leqslant 4U^5$

such that H_ℓ^* (assume ℓ odd) exhibits all corner cuts

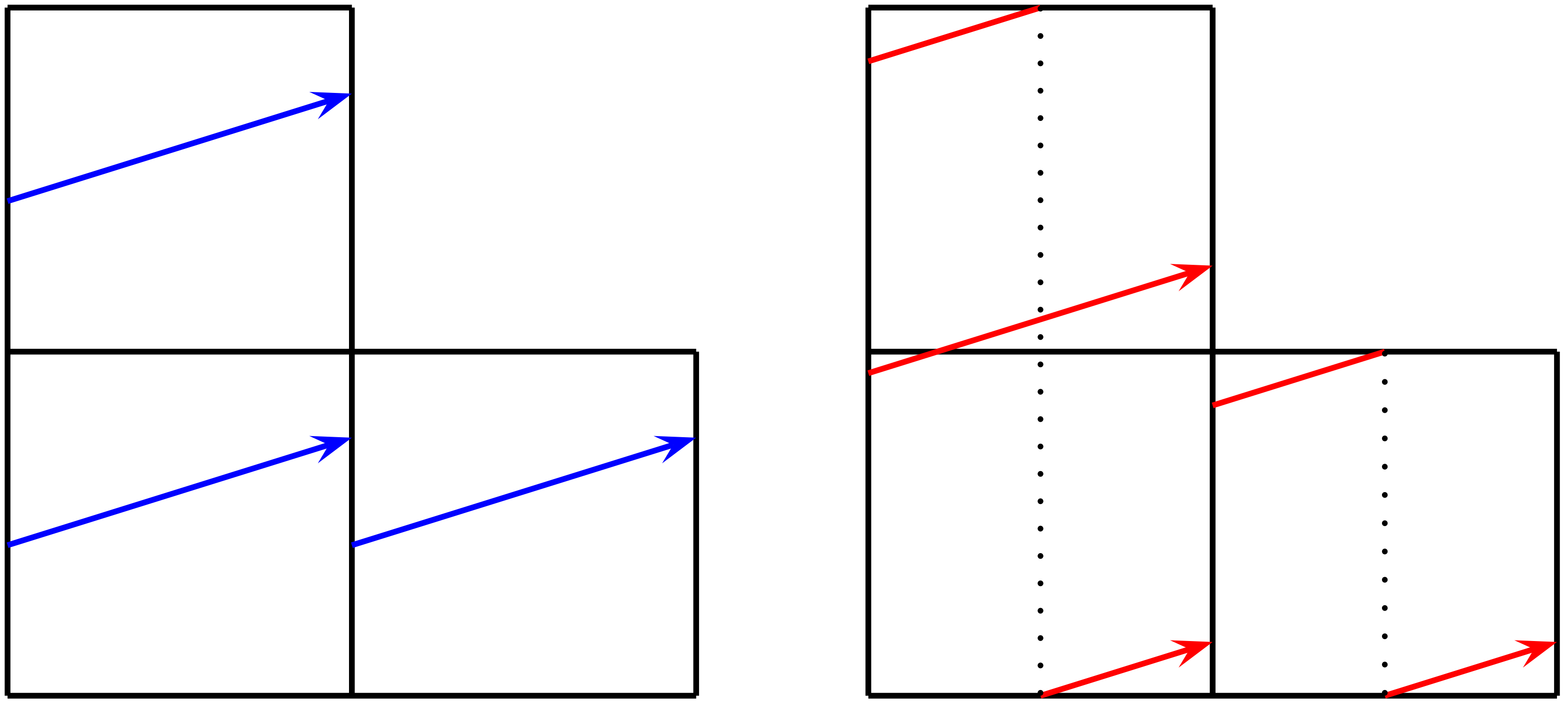




do not give rise to corner cuts

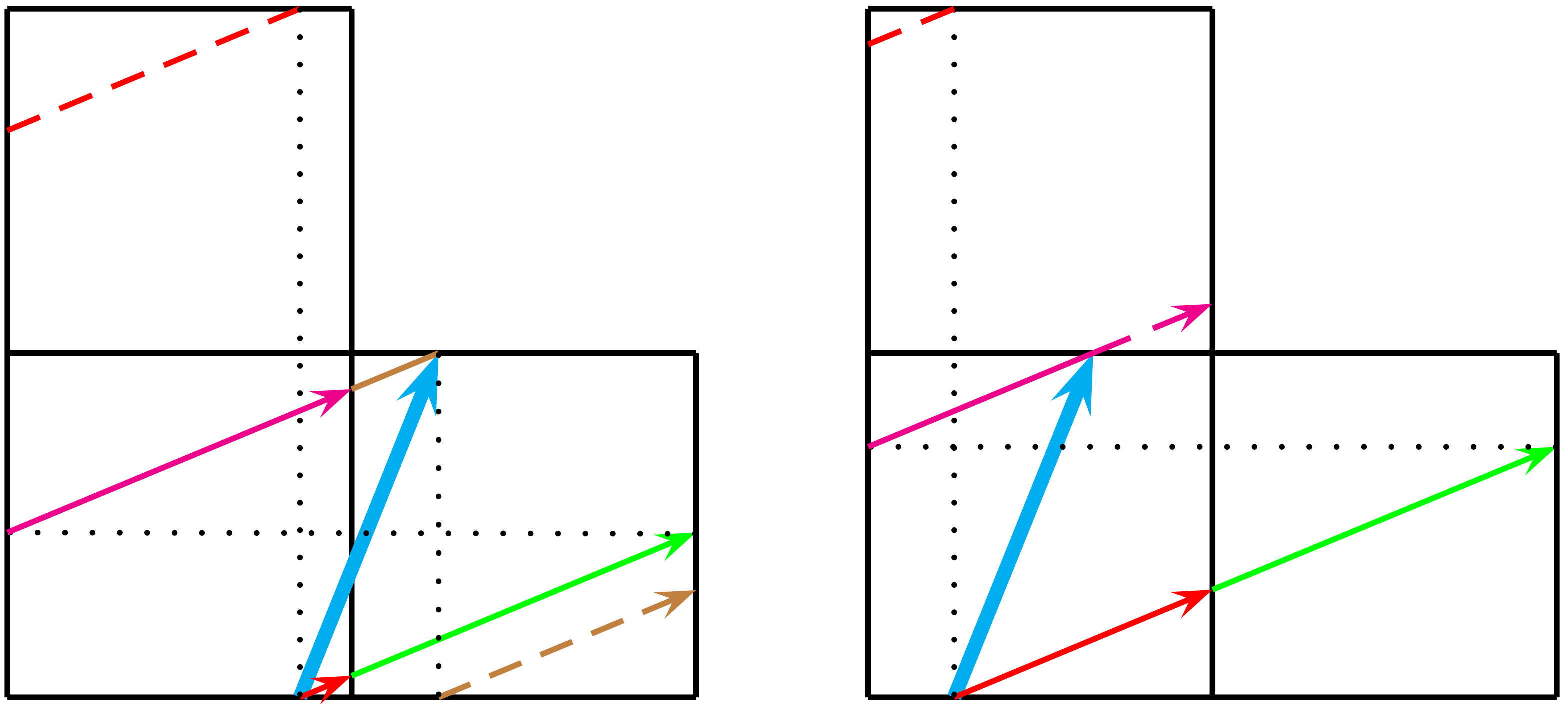
give rise to corner cuts

almost horizontal units and almost vertical units



do not give rise to corner cuts

give rise to corner cuts

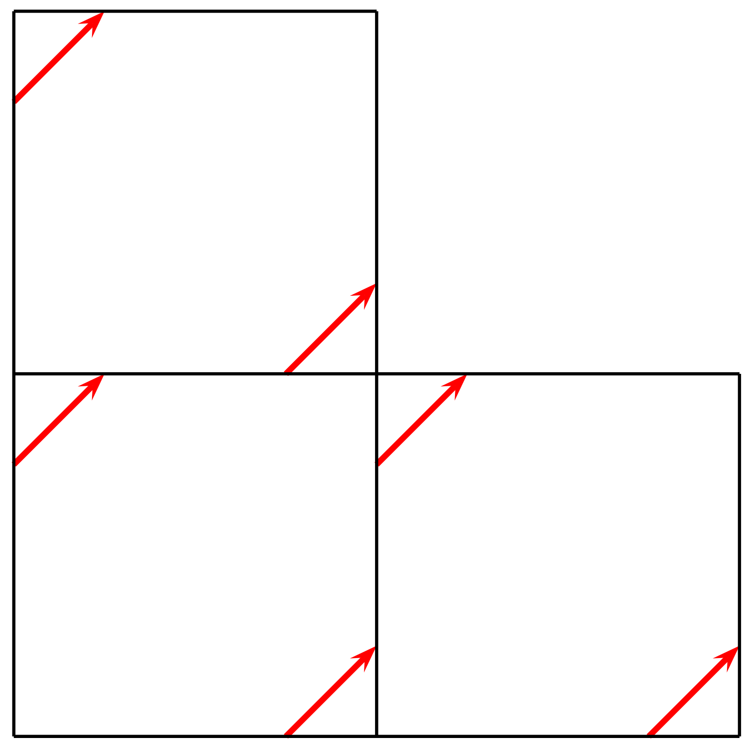


almost vertical unit

ancestor almost horizontal units – possibly fractional at the two ends

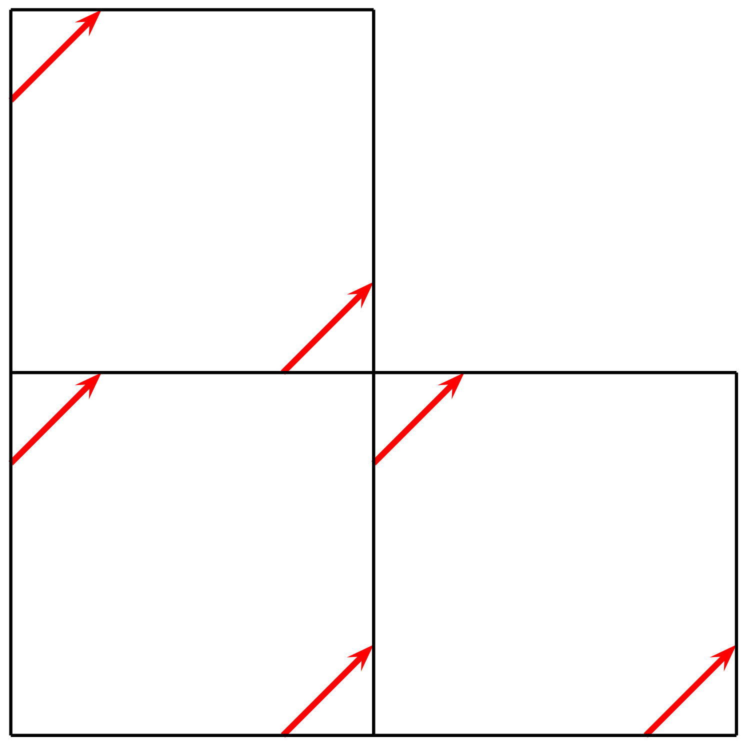
claim − there is some m_ℓ satisfying $2U + 1 \leqslant m_\ell \leqslant 4U^5$

such that H_ℓ^* (assume ℓ odd) exhibits all corner cuts

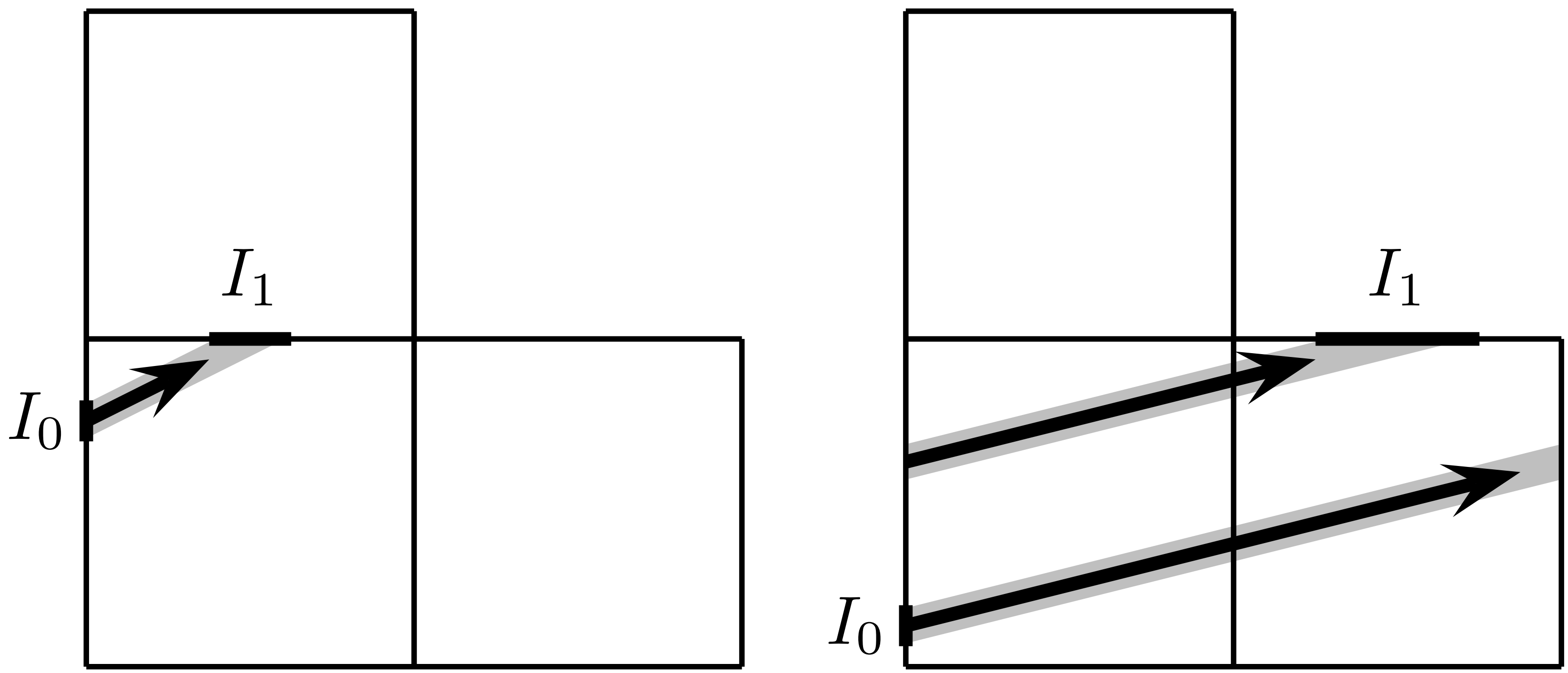


claim − there is some m_ℓ satisfying $2U + 1 \leqslant m_\ell \leqslant 4U^5$

such that H_ℓ^* (assume ℓ odd) exhibits all corner cuts



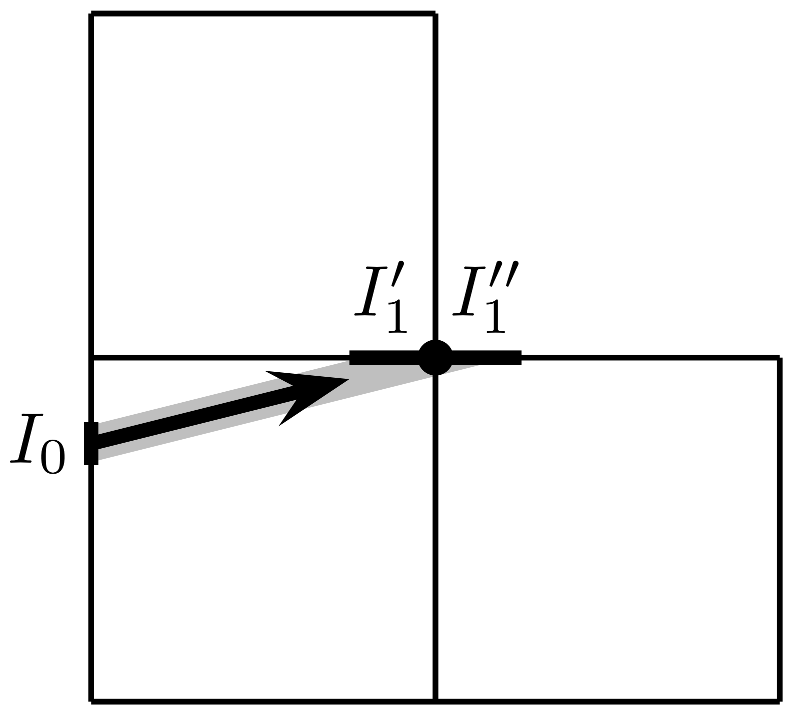
can be established by three rounds of ancestor process area magnification of intervals by α_1^{-1}-flow area magnification of intervals by α_1^{-1}-flow



good – does not hit a singularity

length$(I_1) = \alpha_1$ length(I_0)

bad – hits a singularity (top right vertex of a square face)



I_0 – open V^*-free interval on a vertical edge of a square face

claim – $\text{length}(I_0) \leqslant \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

I_0 – open V^*-free interval on a vertical edge of a square face

claim – $\text{length}(I_0) \leqslant \dfrac{2}{\alpha_1 \dots \alpha_\ell}$

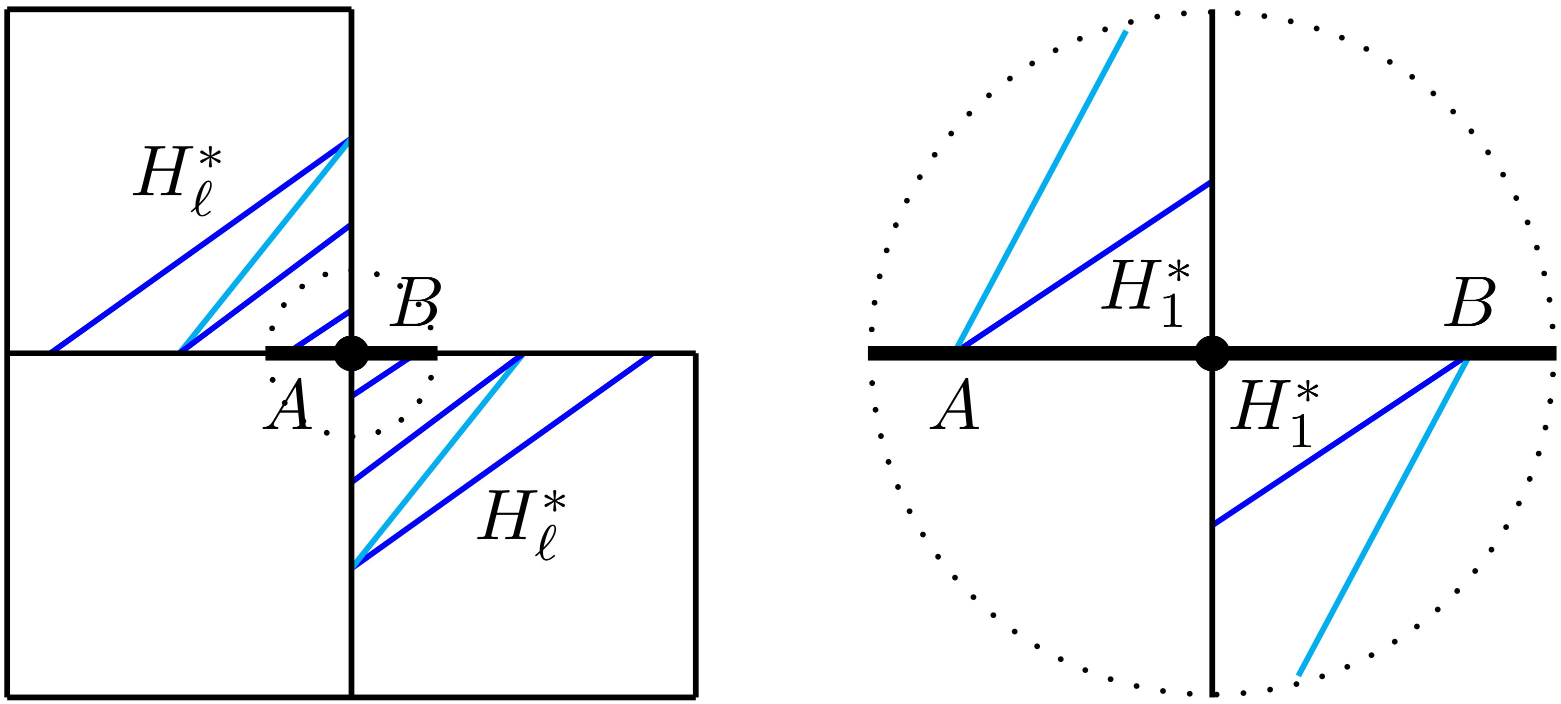
suppose – $\text{length}(I_0) > \dfrac{2}{\alpha_1 \dots \alpha_\ell}$

I_0 − open V^*-free interval on a vertical edge of a square face

claim − $\text{length}(I_0) \leqslant \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

suppose − $\text{length}(I_0) > \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

exponentially fast zigzagging to a street corner



I_0 – open V^*-free interval on a vertical edge of a square face

claim – length$(I_0) \leqslant \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

suppose – length$(I_0) > \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

exponentially fast zigzagging to a street corner gives

I_1 bad $\Rightarrow$ I_1 not H_1^*-free

I_0 – open V^*-free interval on a vertical edge of a square face

claim – length$(I_0) \leqslant \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

suppose – length$(I_0) > \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

exponentially fast zigzagging to a street corner gives

I_1 bad $\Rightarrow$ I_1 not H_1^*-free

vertical edge cutting $\Rightarrow$ I_0 H_1^*-free $\Rightarrow$ I_1 H_1^*-free

I_0 – open V^*-free interval on a vertical edge of a square face

claim – length$(I_0) \leqslant \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$ suppose – length$(I_0) > \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

exponentially fast zigzagging to a street corner gives

I_1 bad $\Rightarrow$ I_1 not H_1^*-free

vertical edge cutting $\Rightarrow$ I_0 H_1^*-free $\Rightarrow$ I_1 H_1^*-free

so I_1 good and H_1^*-free – length$(I_1) = \alpha_1$ length$(I_0) > \dfrac{2}{\alpha_2 \ldots \alpha_\ell}$

I_0 V^*-free – length$(I_0) > \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

$\Rightarrow$ I_1 good and H_1^*-free – length$(I_1) > \dfrac{2}{\alpha_2 \ldots \alpha_\ell}$

I_0 V^*-free – length$(I_0) > \dfrac{2}{\alpha_1 \dots \alpha_\ell}$

$\Rightarrow$ I_1 good and H_1^*-free – length$(I_1) > \dfrac{2}{\alpha_2 \dots \alpha_\ell}$

$\Rightarrow$ I_2 good and V_2^*-free – length$(I_2) > \dfrac{2}{\alpha_3 \dots \alpha_\ell}$

$\Rightarrow$ ...

I_0 V^*-free – length(I_0) $> \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

$\Rightarrow$ I_1 good and H_1^*-free – length(I_1) $> \dfrac{2}{\alpha_2 \ldots \alpha_\ell}$

$\Rightarrow$ I_2 good and V_2^*-free – length(I_2) $> \dfrac{2}{\alpha_3 \ldots \alpha_\ell}$

$\Rightarrow \ldots$

$\Rightarrow$ $I_{\ell-1}$ good and $V_{\ell-1}^*$-free – length($I_{\ell-1}$) $> \dfrac{2}{\alpha_\ell}$

I_0 V^*-free – $\text{length}(I_0) > \dfrac{2}{\alpha_1 \ldots \alpha_\ell}$

$\Rightarrow I_1$ good and H_1^*-free – $\text{length}(I_1) > \dfrac{2}{\alpha_2 \ldots \alpha_\ell}$

$\Rightarrow I_2$ good and V_2^*-free – $\text{length}(I_2) > \dfrac{2}{\alpha_3 \ldots \alpha_\ell}$

$\Rightarrow \ldots$

$\Rightarrow I_{\ell-1}$ good and $V_{\ell-1}^*$-free – $\text{length}(I_{\ell-1}) > \dfrac{2}{\alpha_\ell}$

but then $\alpha_\ell \, \text{length}(I_{\ell-1}) > 2 \Rightarrow I_\ell$ contains whole edge of square face

$\Rightarrow I_\ell$ not H_ℓ^*-free $\Rightarrow\Leftarrow$

I_0 – open V^*-free interval on a vertical edge of a square face

$$\text{length}(I_0) \leqslant \frac{2}{\alpha_1 \dots \alpha_\ell}$$

I_0 – open V^*-free interval on a vertical edge of a square face

$$\text{length}(I_0) \leqslant \frac{2}{\alpha_1 \dots \alpha_\ell}$$

$$m_0 = m_\ell \alpha_1 \dots \alpha_\ell$$

$$m_\ell \leqslant 4U^5$$

$$\alpha_1, \dots, \alpha_\ell < U$$

$$\text{length}(V^*) = m_0(1+\alpha^2)^{1/2} = m_\ell \alpha_1 \dots \alpha_\ell (1+\alpha^2)^{1/2}$$

I_0 – open V^*-free interval on a vertical edge of a square face

$$\text{length}(I_0) \leqslant \frac{2}{\alpha_1 \ldots \alpha_\ell}$$

$$m_0 = m_\ell \alpha_1 \ldots \alpha_\ell$$

$$m_\ell \leqslant 4U^5$$

$$\alpha_1, \ldots, \alpha_\ell < U$$

$$\text{length}(V^*) = m_0(1+\alpha^2)^{1/2} = m_\ell \alpha_1 \ldots \alpha_\ell (1+\alpha^2)^{1/2}$$

$$\frac{2}{\alpha_1 \ldots \alpha_\ell} = \frac{2m_\ell(1+\alpha^2)^{1/2}}{\text{length}(V^*)}$$

I_0 – open V^*-free interval on a vertical edge of a square face

$$\text{length}(I_0) \leqslant \frac{2}{\alpha_1 \dots \alpha_\ell}$$

$$m_0 = m_\ell \alpha_1 \dots \alpha_\ell$$

$$m_\ell \leqslant 4U^5$$

$$\alpha_1, \dots, \alpha_\ell < U$$

$$\text{length}(V^*) = m_0(1+\alpha^2)^{1/2} = m_\ell \alpha_1 \dots \alpha_\ell (1+\alpha^2)^{1/2}$$

$$\frac{2}{\alpha_1 \dots \alpha_\ell} = \frac{2m_\ell(1+\alpha^2)^{1/2}}{\text{length}(V^*)}$$

$$\text{length}(I_0) \leqslant \frac{12U^5}{\text{length}(V^*)} \hookrightarrow \text{superdensity}$$

$\mathcal{P}$ – finite polysquare surface

h – street-LCM of $\mathcal{P}$ – LCM of all horizontal and vertical street lengths

$\mathcal{P}$ finite $\Rightarrow$ h well defined and finite

$\mathcal{P}$ – finite polysquare surface

h – street-LCM of $\mathcal{P}$ – LCM of all horizontal and vertical street lengths

$\mathcal{P}$ finite $\Rightarrow$ h well defined and finite

Beck–C–Yang ($\geqslant$ 2020)

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$ badly approximable

where $a_0, a_1, a_2, a_3, \ldots \in \mathbb{N}$ are multiples of h and bounded

$\Rightarrow$ any V with slope α is superdense on finite polysquare surface $\mathcal{P}$

$\mathcal{P}$ – finite polysquare surface

h – street-LCM of $\mathcal{P}$ – LCM of all horizontal and vertical street lengths

$\mathcal{P}$ finite $\Rightarrow$ h well defined and finite

Beck–C–Yang ($\geqslant$ 2020)

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$ badly approximable

where $a_0, a_1, a_2, a_3, \ldots \in \mathbb{N}$ are multiples of h and bounded

$\Rightarrow$ any V with slope α is superdense on finite polysquare surface $\mathcal{P}$

Beck–C ($\geqslant$ 2020)

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$ badly approximable

where $a_0, a_1, a_2, a_3, \ldots \in \mathbb{N}$ are bounded

$\Rightarrow$ any V with slope α is superdense on finite polysquare surface $\mathcal{P}$

Beck–C ($\geqslant$ 2020)

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$ badly approximable

where $a_0, a_1, a_2, a_3, \ldots \in \mathbb{N}$ are bounded

$\Rightarrow$ any V with slope α is superdense on finite polysquare surface $\mathcal{P}$

same for any billiard orbit with initial slope α in finite polysquare region

Beck–C ($\geqslant$ 2020) (to be checked carefully again)

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$ badly approximable

where $a_0, a_1, a_2, a_3, \ldots \in \mathbb{N}$ are bounded

$\Rightarrow$ any V with slope α is superdense on finite polysquare surface $\mathcal{P}$

same for any billiard orbit with initial slope α in finite polysquare region

finite or infinite polysquare surface with street length at most ℓ

finite or infinite polysquare surface with street length at most ℓ

Beck–C–Yang ($\geqslant$ 2020)

$$\alpha = [a; a, a, a, \ldots] = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cdots}}}$$

where $a \in \mathbb{N}$ is multiple of ℓ!

$\Rightarrow$ any V from vertex and with slope α is dense on ℓ-square-maze $\mathcal{P}$

finite or infinite polysquare surface with street length at most ℓ

Beck–C–Yang ($\geqslant$ 2020)

$$\alpha = [a; a, a, a, \ldots] = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cdots}}}$$

where $a \in \mathbb{N}$ is multiple of ℓ!

$\Rightarrow$ any V from vertex and with slope α is dense on ℓ-square-maze $\mathcal{P}$

for any square face S_0 of $\mathcal{P}$, there exists constant $c_0 = c_0(S_0; \varepsilon; \alpha)$

$n \geqslant c_0$, $Q \in S_0 \Rightarrow$ initial segment of V of length $n^{3+\varepsilon}$ is $1/n$-close to Q

finite or infinite polysquare surface with street length at most ℓ

Beck–C–Yang ($\geqslant$ 2020)

$$\alpha = [a; a, a, a, \ldots] = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cdots}}}$$

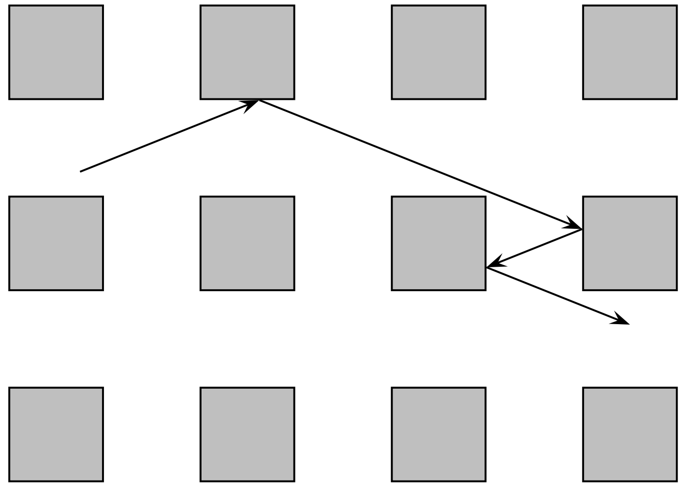
where $a \in \mathbb{N}$ is multiple of ℓ!

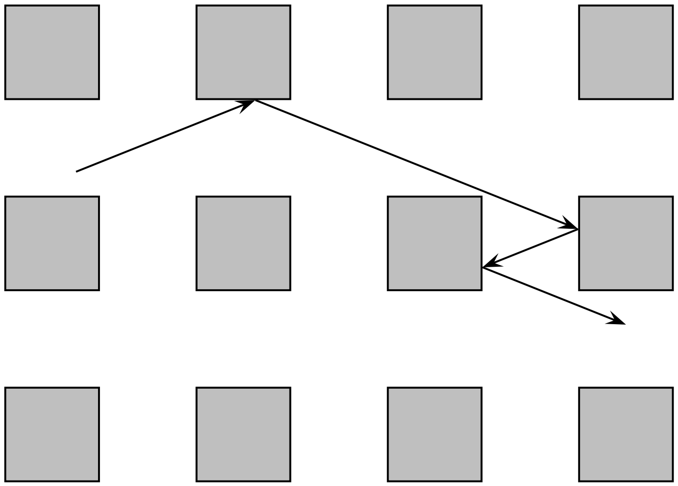
$\Rightarrow$ any V from vertex and with slope α is dense on ℓ-square-maze $\mathcal{P}$

for any square face S_0 of $\mathcal{P}$, there exists constant $c_0 = c_0(S_0; \varepsilon; \alpha)$

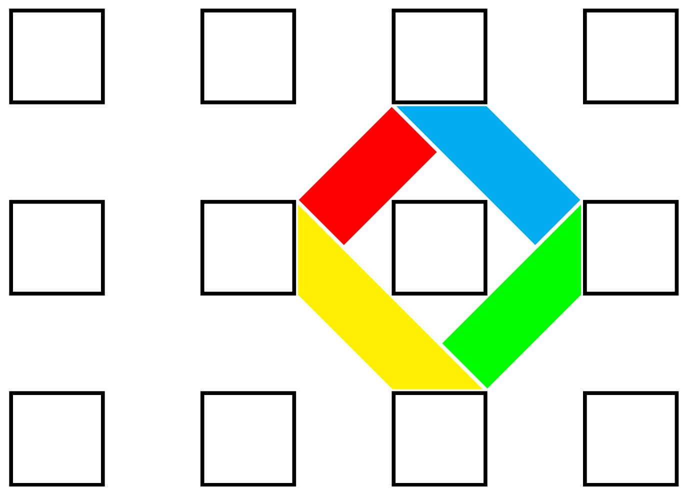
$n \geqslant c_0$, $Q \in S_0$ $\Rightarrow$ initial segment of V of length $n^{3+\varepsilon}$ is $1/n$-close to Q

not superdensity but still time-quantitative density

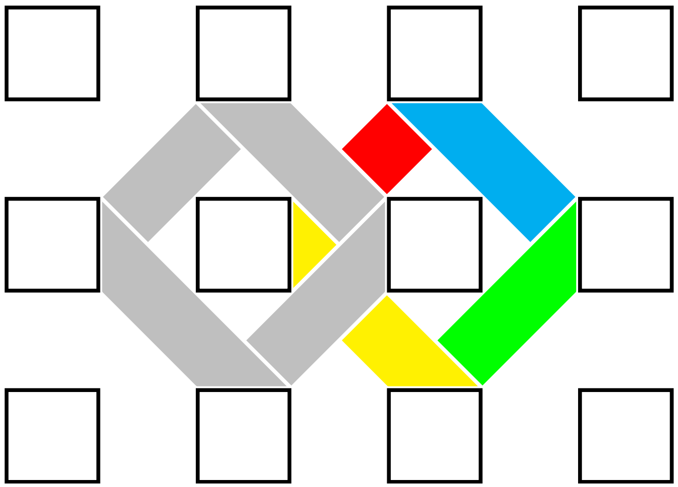




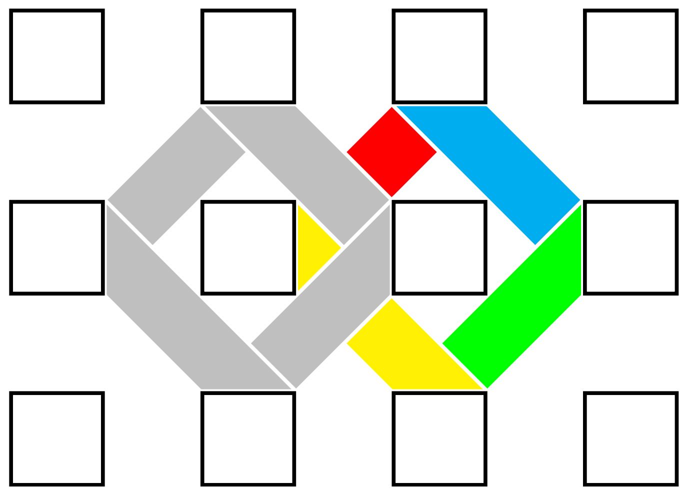
horizontal and vertical streets have unbounded lengths a finite tilted street



a finite tilted street and a perpendicular finite tilted street

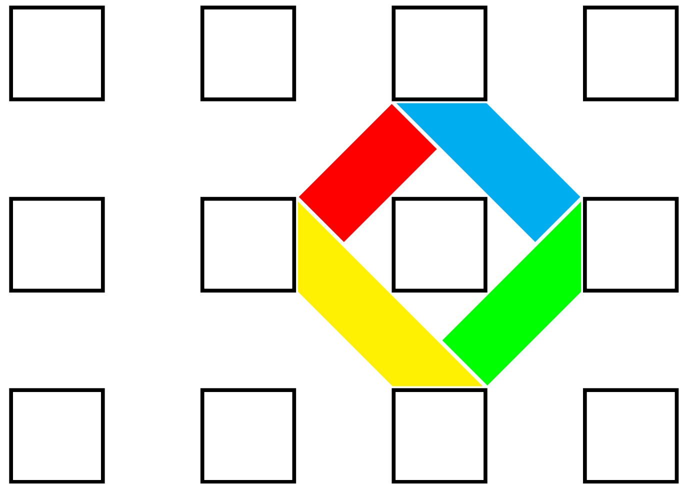


a finite tilted street and a perpendicular finite tilted street

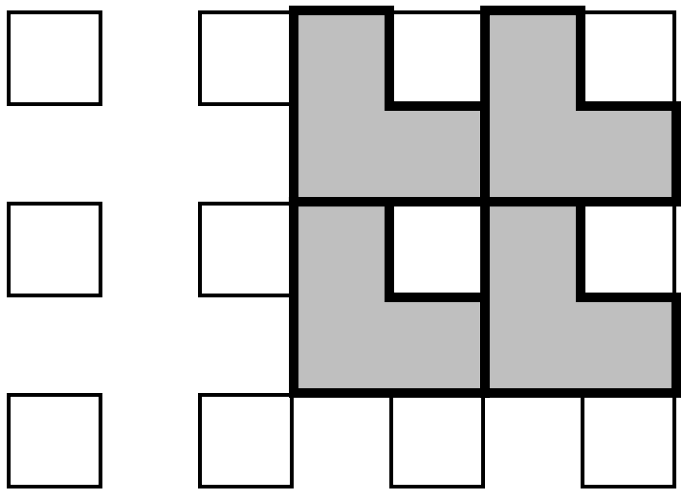


square-maze if we look NE-SW and NW-SE

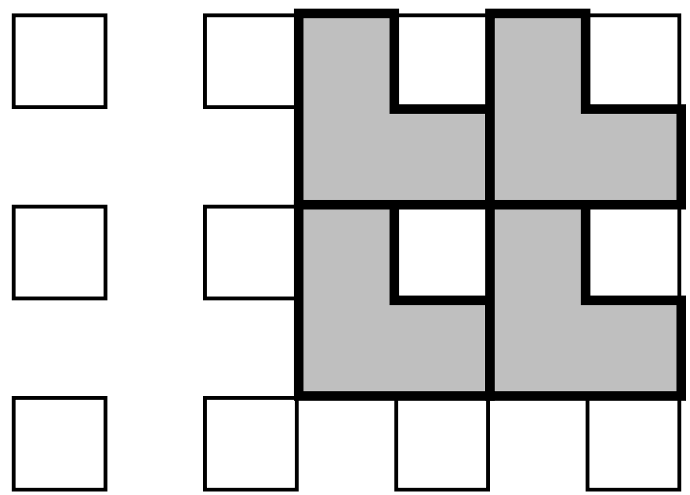
a finite tilted street



a finite tilted street is contained in 4 L-shapes

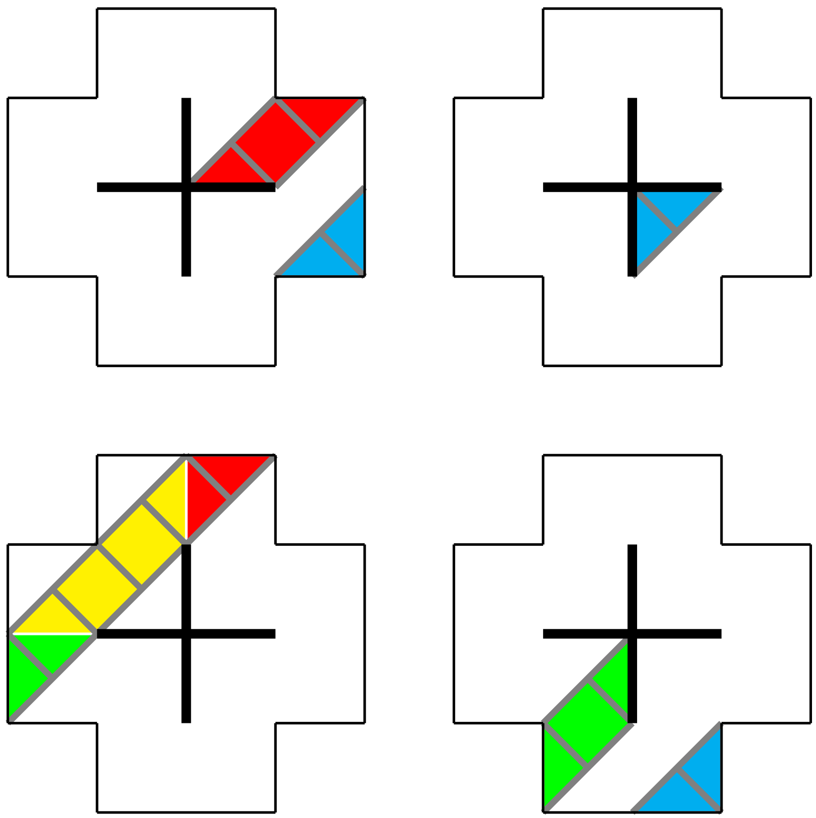


a finite tilted street is contained in 4 L-shapes

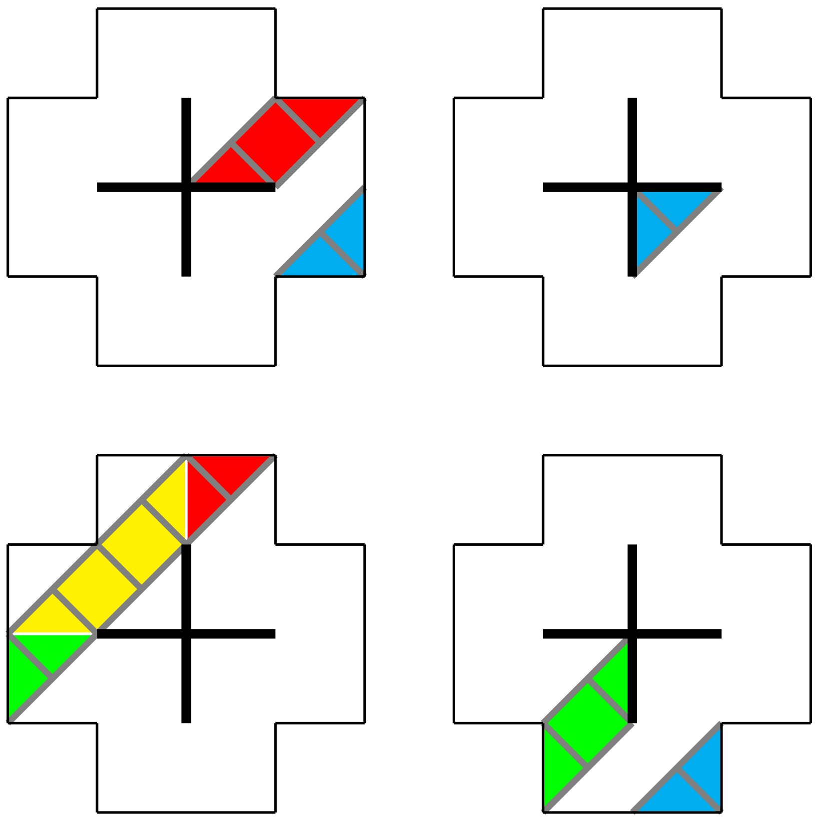


unfolding $\hookrightarrow$ street contained in 4 L-crosses glued together suitably

unfolding $\hookrightarrow$ street contained in 4 L-crosses glued together suitably



unfolding $\hookrightarrow$ street contained in 4 L-crosses glued together suitably

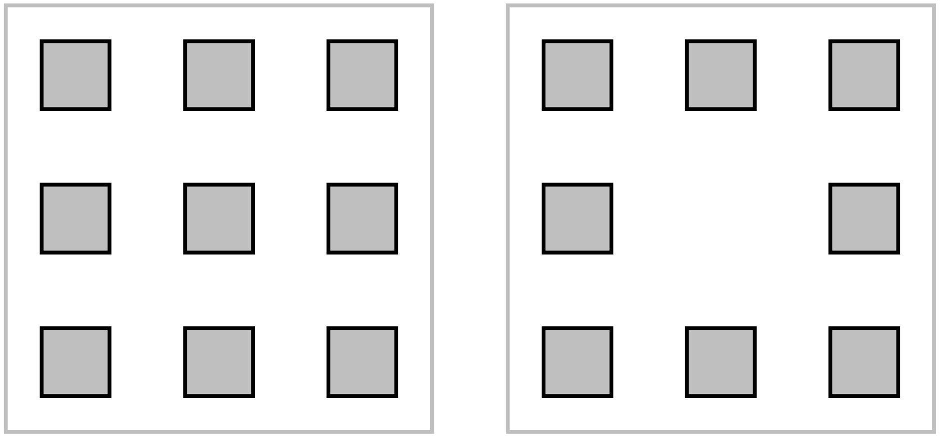


12-square-maze

split the periodic wind-tree model into 3×3 blocks of obstacles aperiodic modifications of periodic wind-tree model

split the periodic wind-tree model into 3×3 blocks of obstacles

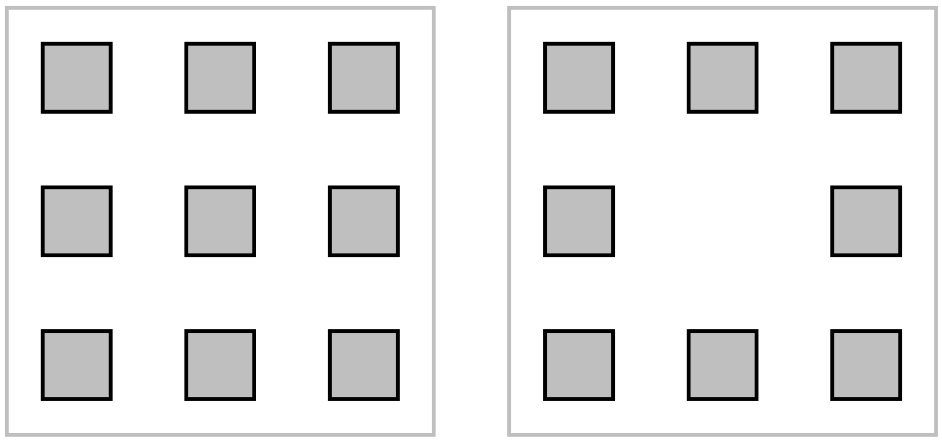
modify some blocks by removing the middle obstacle



aperiodic modifications of periodic wind-tree model

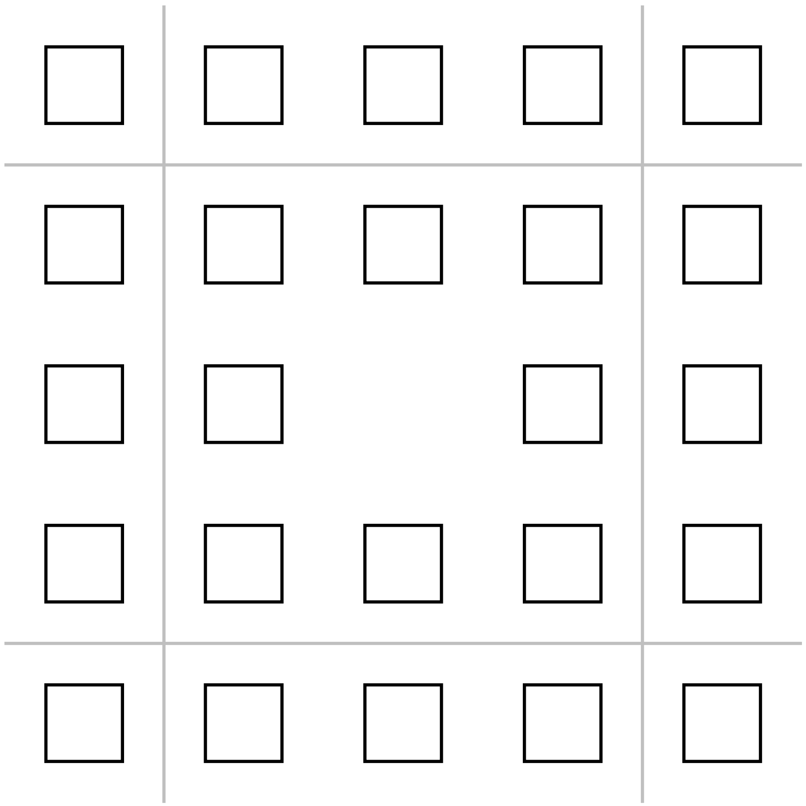
split the periodic wind-tree model into 3×3 blocks of obstacles

modify some blocks by removing the middle obstacle



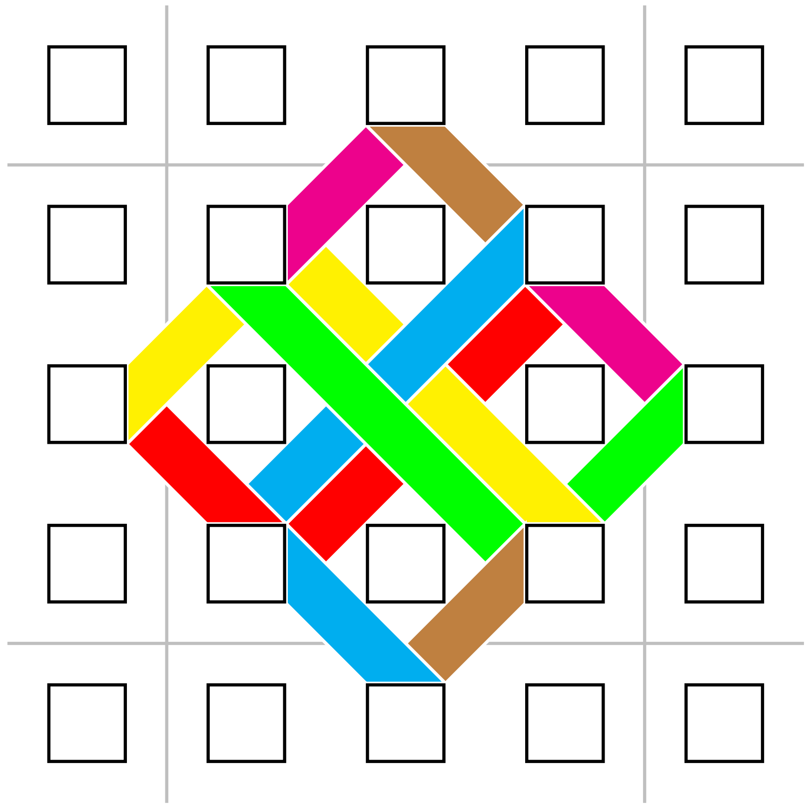
$2^{\aleph_0} = c$-many aperiodic ways of doing this

take modified 3×3 block



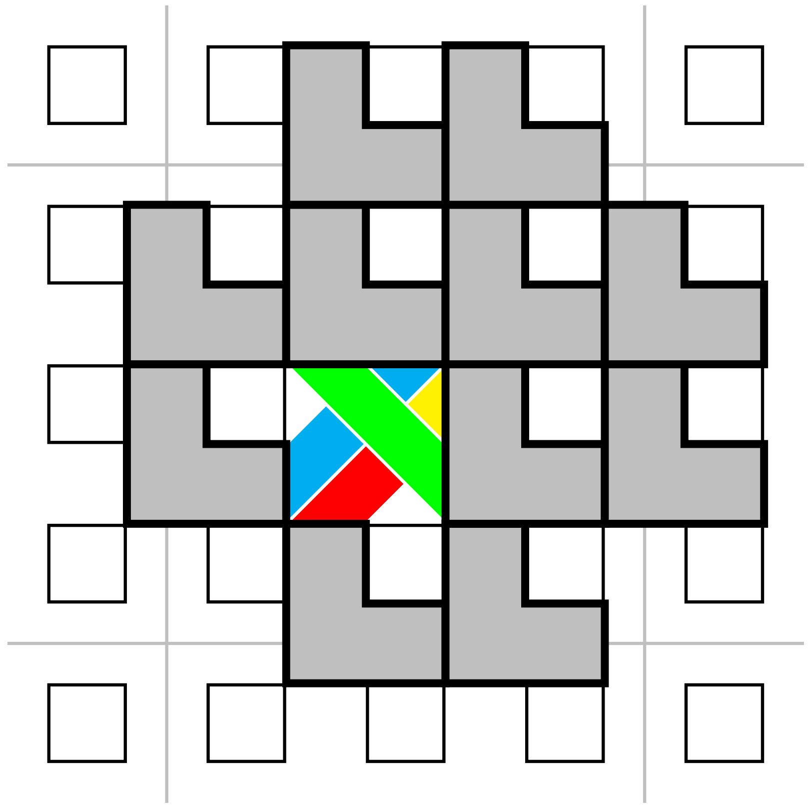
surrounded by 15 more obstacles

take modified 3×3 block



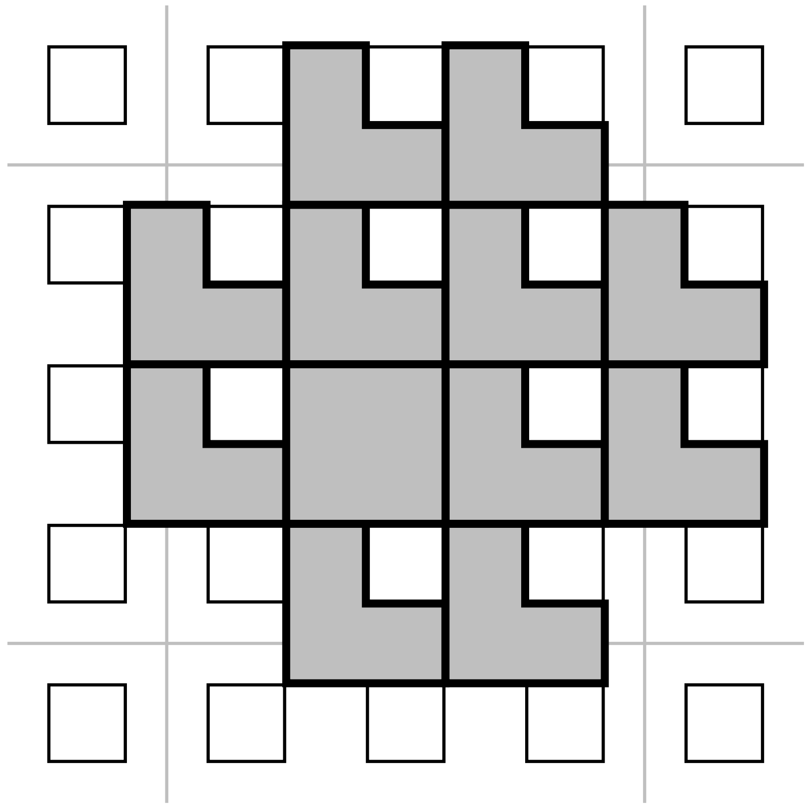
a tilted street

take modified 3×3 block



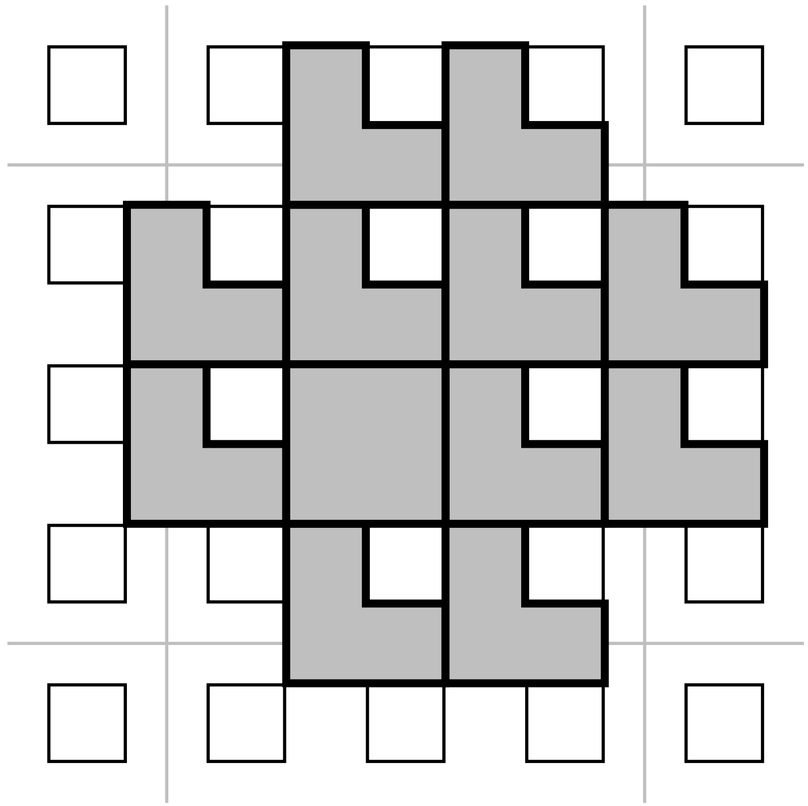
covered by 11 L-shapes

take modified 3×3 block



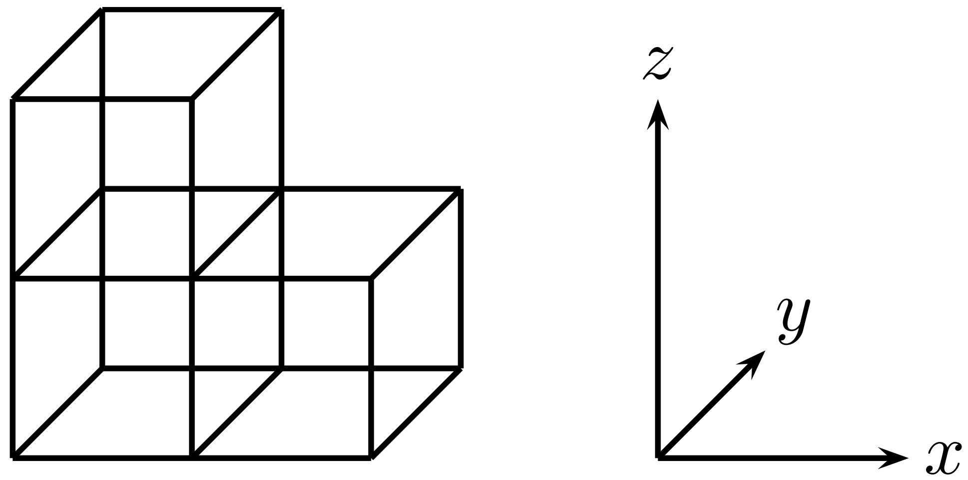
covered by 11 L-shapes and 1 square

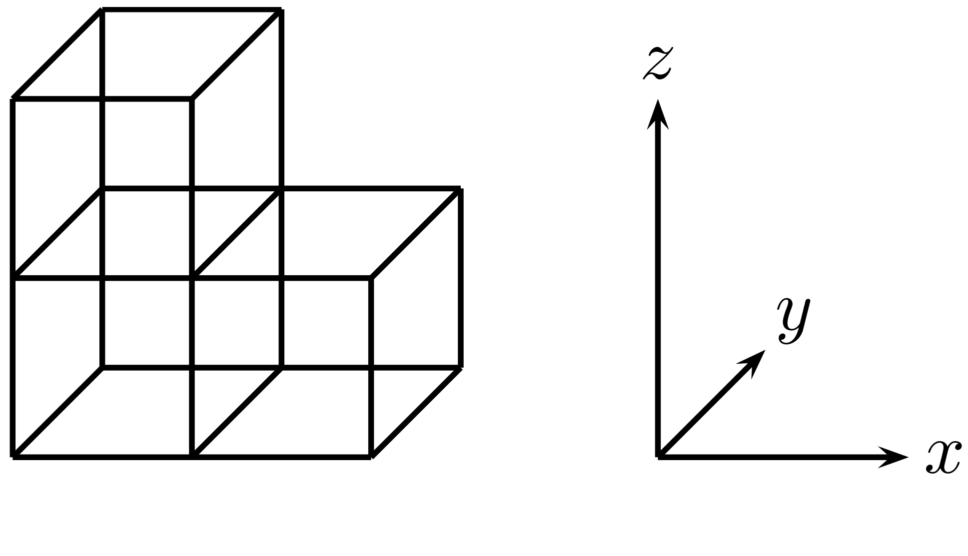
take modified 3×3 block

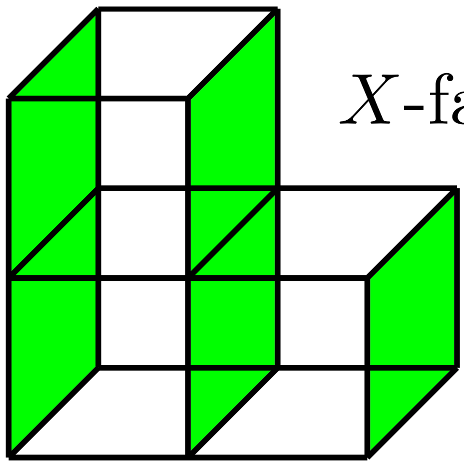


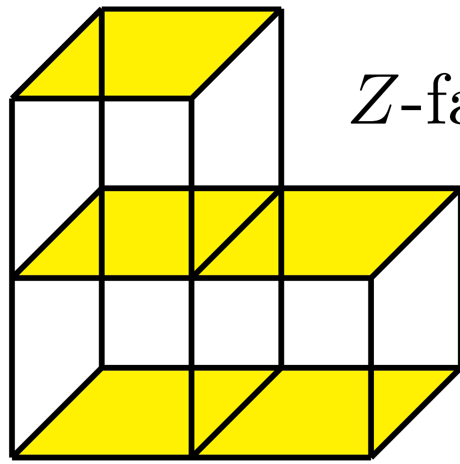
covered by 11 L-shapes and 1 square

unfolding $\hookrightarrow$ ℓ-square-maze for some finite ℓ

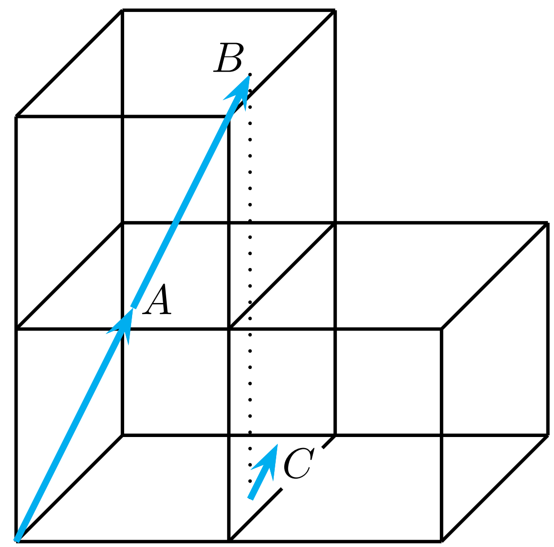

z
y
x


z
y
x

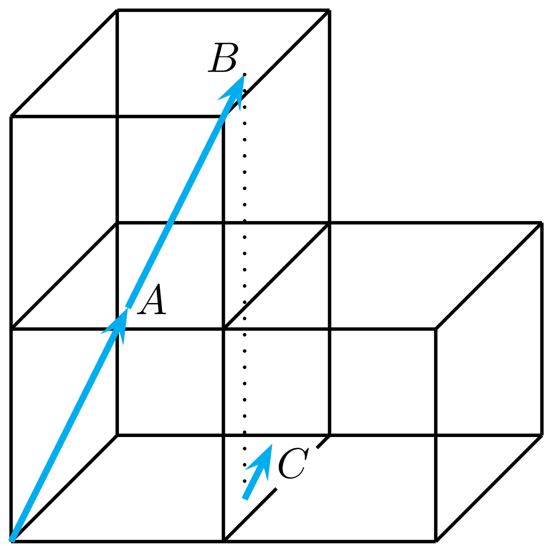

X-faces


Z-faces

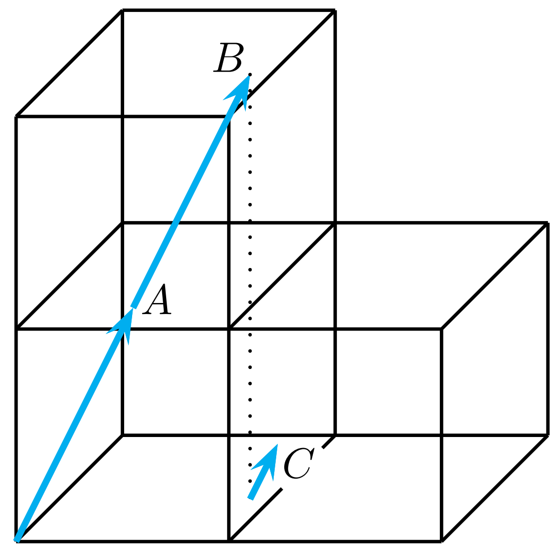
1-direction geodesic on L-solid manifold



1-direction geodesic on L-solid manifold



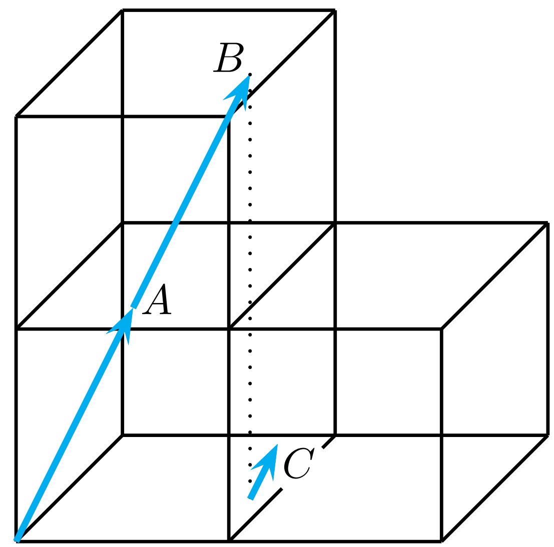
$$A = (x, y, 1),\ B = (2kx, 2ky, 2),\ B' = (2kx, 2ky, 0),\ C = (1, x, y)$$



$A = (x, y, 1)$, $B = (2kx, 2ky, 2)$, $B' = (2kx, 2ky, 0)$, $C = (1, x, y)$

$\mathbf{0}A$ and $B'C$ parallel $\hookrightarrow \dfrac{1-2kx}{x} = \dfrac{x-2ky}{y} = \dfrac{y-0}{1}$

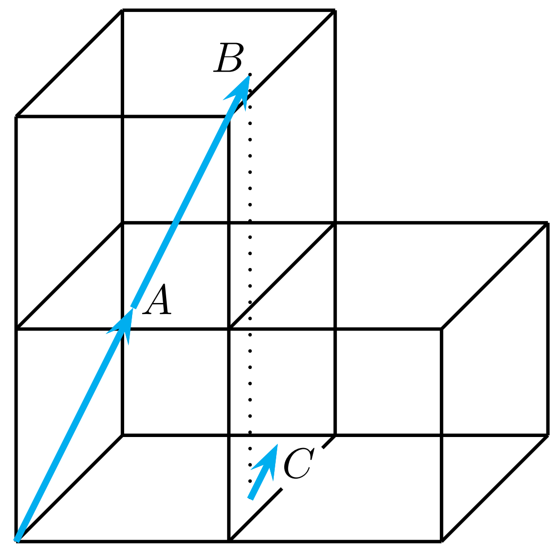
$y = x^2$ and $x^3 + 2kx - 1 = 0$



$A = (x, y, 1),\ B = (2kx, 2ky, 2),\ B' = (2kx, 2ky, 0),\ C = (1, x, y)$

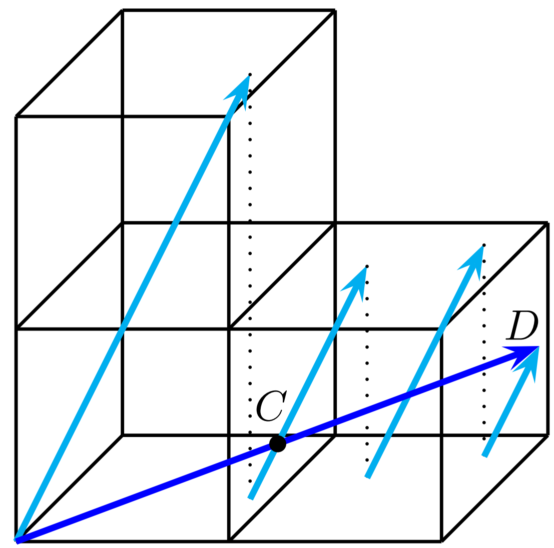
$\mathbf{0}A$ and $B'C$ parallel $\hookrightarrow \dfrac{1-2kx}{x} = \dfrac{x-2ky}{y} = \dfrac{y-0}{1}$

$y = x^2$ and $x^3 + 2kx - 1 = 0 \hookrightarrow$ root α_k satisfying $\dfrac{1}{2k+1} < \alpha_k < \dfrac{1}{2k}$



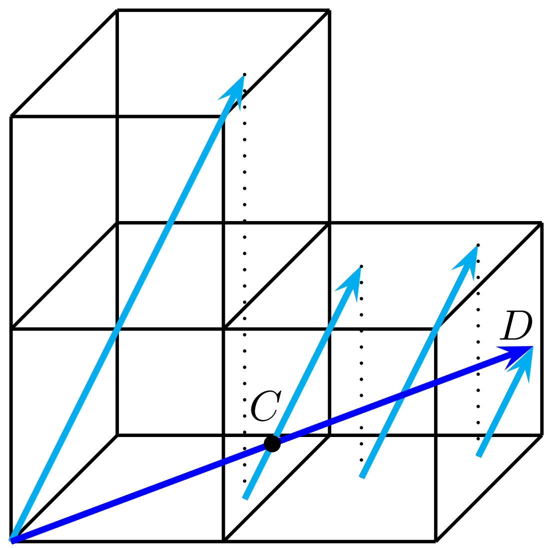
$A = (x, y, 1)$, $B = (2kx, 2ky, 2)$, $B' = (2kx, 2ky, 0)$, $C = (1, x, y)$

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$



$A = (x, y, 1)$, $B = (2kx, 2ky, 2)$, $B' = (2kx, 2ky, 0)$, $C = (1, x, y)$

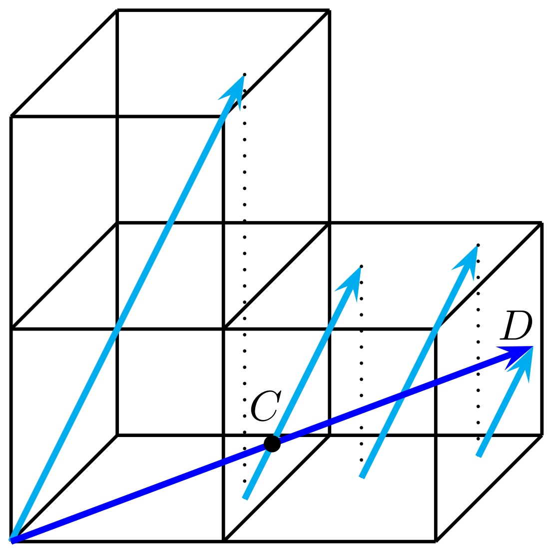
geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$



$A = (x, y, 1)$, $B = (2kx, 2ky, 2)$, $B' = (2kx, 2ky, 0)$, $C = (1, x, y)$

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$

shortline $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$



$A = (x, y, 1)$, $B = (2kx, 2ky, 2)$, $B' = (2kx, 2ky, 0)$, $C = (1, x, y)$

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$

shortline $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$

same X-face hitting

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$

shortline $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$ X-face hitting

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$

shortline $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$ — X-face hitting

geodesic $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$

shortline $\mathcal{L}_k^{**}$ has direction vector $\mathbf{v}_2 = (\alpha_k^2, 1, \alpha_k)$ — Y-face hitting

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$

shortline $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$ X-face hitting

geodesic $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$

shortline $\mathcal{L}_k^{**}$ has direction vector $\mathbf{v}_2 = (\alpha_k^2, 1, \alpha_k)$ Y-face hitting

geodesic $\mathcal{L}_k^{**}$ has direction vector $\mathbf{v}_2 = (\alpha_k^2, 1, \alpha_k)$

shortline has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ Z-face hitting

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$

shortline $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$ X-face hitting

geodesic $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$

shortline $\mathcal{L}_k^{**}$ has direction vector $\mathbf{v}_2 = (\alpha_k^2, 1, \alpha_k)$ Y-face hitting

geodesic $\mathcal{L}_k^{**}$ has direction vector $\mathbf{v}_2 = (\alpha_k^2, 1, \alpha_k)$

shortline $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ Z-face hitting

geodesic $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$

shortline $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$ X-face hitting

geodesic $\mathcal{L}_k^*$ has direction vector $\mathbf{v}_1 = (1, \alpha_k, \alpha_k^2)$

shortline $\mathcal{L}_k^{**}$ has direction vector $\mathbf{v}_2 = (\alpha_k^2, 1, \alpha_k)$ Y-face hitting

geodesic $\mathcal{L}_k^{**}$ has direction vector $\mathbf{v}_2 = (\alpha_k^2, 1, \alpha_k)$

shortline $\mathcal{L}_k$ has direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ Z-face hitting

shortline process $\mathcal{L}_k \longrightarrow \mathcal{L}_k^* \longrightarrow \mathcal{L}_k^{**} \longrightarrow \mathcal{L}_k \longrightarrow \ldots$

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – L-solid manifold

root α_k of $x^3 + 2kx - 1 = 0$ satisfying $\dfrac{1}{2k+1} < \alpha_k < \dfrac{1}{2k}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is dense in $\mathcal{P}$

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – L-solid manifold

root α_k of $x^3 + 2kx - 1 = 0$ satisfying $\dfrac{1}{2k+1} < \alpha_k < \dfrac{1}{2k}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is dense in $\mathcal{P}$

there exist constants $k_0 = k_0(\varepsilon)$ and $c_0 = c_0(\varepsilon; \mathbf{v}_0)$

$k \geqslant k_0$, $n \geqslant c_0$, $Q \in \mathcal{P}$

$\Rightarrow$ initial segment of $\mathcal{L}_{\mathbf{v}_0}$ of length $n^{2+\varepsilon}$ is $1/n$-close to Q

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – L-solid manifold

root α_k of $x^3+2kx-1=0$ satisfying $\dfrac{1}{2k+1}<\alpha_k<\dfrac{1}{2k}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0=(\alpha_k,\alpha_k^2,1)$ is dense in $\mathcal{P}$

there exist constants $k_0=k_0(\varepsilon)$ and $c_0=c_0(\varepsilon;\mathbf{v}_0)$

$k\geqslant k_0$, $n\geqslant c_0$, $Q\in\mathcal{P}$

$\Rightarrow$ initial segment of $\mathcal{L}_{\mathbf{v}_0}$ of length $n^{2+\varepsilon}$ is $1/n$-close to Q

not superdensity but still time-quantitative density

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – finite polycube manifold with street-LCM h

root α_k of $x^3 + hkx - 1 = 0$ satisfying $\dfrac{1}{hk+1} < \alpha_k < \dfrac{1}{hk}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is dense in $\mathcal{P}$

there exist constants $k_0 = k_0(\varepsilon)$ and $c_0 = c_0(\varepsilon; \mathbf{v}_0)$

$k \geqslant k_0$, $n \geqslant c_0$, $Q \in \mathcal{P}$

$\Rightarrow$ initial segment of $\mathcal{L}_{\mathbf{v}_0}$ of length $n^{2+\varepsilon}$ is $1/n$-close to Q

$\hookrightarrow$ a result on uniformity

Beck–C ($\geqslant$ 2020)

$\mathcal{P}$ – finite polycube manifold with street-LCM h

root α_k of $x^3 + hkx - 1 = 0$ satisfying $\dfrac{1}{hk+1} < \alpha_k < \dfrac{1}{hk}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is uniform in $\mathcal{P}$

test sets – all Jordan measurable subsets of $\mathcal{P}$

$\hookrightarrow$ a result on uniformity

Beck–C ($\geqslant$ 2020)

$\mathcal{P}$ – finite polycube manifold with street-LCM h

root α_k of $x^3 + hkx - 1 = 0$ satisfying $\dfrac{1}{hk+1} < \alpha_k < \dfrac{1}{hk}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is uniform in $\mathcal{P}$

test sets – all Jordan measurable subsets of $\mathcal{P}$

time-quantitative result on a weaker form of uniformity – lower bound

$\hookrightarrow$ a result on uniformity

Beck–C ($\geqslant$ 2020)

$\mathcal{P}$ – finite polycube manifold with street-LCM h

root α_k of $x^3 + hkx - 1 = 0$ satisfying $\dfrac{1}{hk+1} < \alpha_k < \dfrac{1}{hk}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is uniform in $\mathcal{P}$

test sets – all Jordan measurable subsets of $\mathcal{P}$

time-quantitative result on a weaker form of uniformity – lower bound

ergodic theory + shortline method + area magnification

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – ℓ-cube-maze

root α_k of $x^3 + \ell!kx - 1 = 0$ satisfying $\dfrac{1}{\ell!k+1} < \alpha_k < \dfrac{1}{\ell!k}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is dense in $\mathcal{P}$

Beck–C–Yang ($\geqslant$ 2020)

$\mathcal{P}$ – ℓ-cube-maze

root α_k of $x^3 + \ell!kx - 1 = 0$ satisfying $\dfrac{1}{\ell!k+1} < \alpha_k < \dfrac{1}{\ell!k}$ with k large

$\Rightarrow$ any $\mathcal{L}_{\mathbf{v}_0}$ with direction vector $\mathbf{v}_0 = (\alpha_k, \alpha_k^2, 1)$ is dense in $\mathcal{P}$

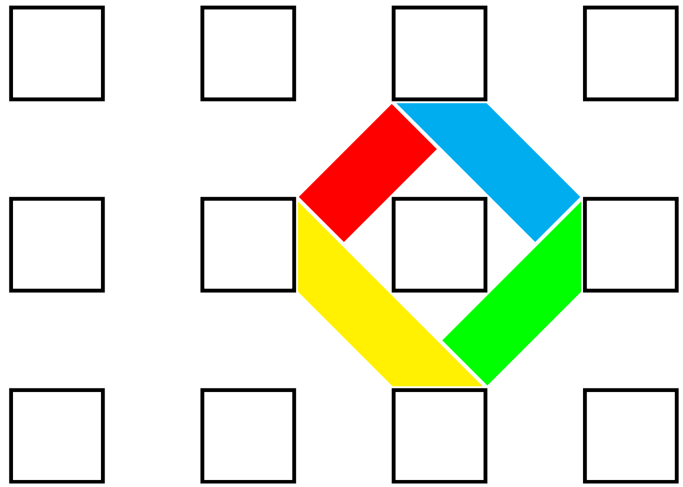
there exists constant $k_0 = k_0(\varepsilon)$

for every cube atom H_0 of $\mathcal{P}$, there exists constant $c_0 = c_0(H_0; \varepsilon; \mathbf{v}_0)$

$k \geqslant k_0$, $n \geqslant c_0$, $Q \in H_0$

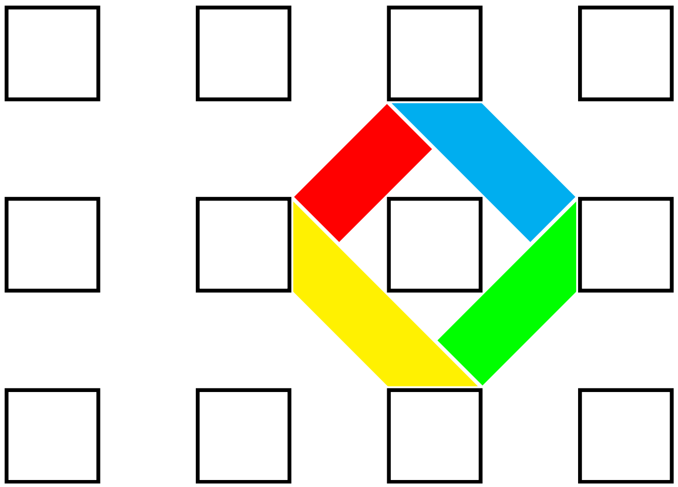
$\Rightarrow$ initial segment of $\mathcal{L}_{\mathbf{v}_0}$ of length $n^{38+\varepsilon}$ is $1/n$-close to Q

no application to 3-dimensional periodic wind-tree model dimension 2



no application to 3-dimensional periodic wind-tree model

dimension 2



no satisfactory analogue in 3 dimensions

THANK YOU