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MMP for surface foliations

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Goal Produce minimal models of foliations on 3-folds

<u>Thm</u> X proj. 3-fold $\mathcal{F}$ corank 1 foliation simple singularities. Then $\exists$ an MMP for $\mathcal{F}$

$$X \overset{\phi}{\dashrightarrow} X' \qquad \phi \text{ bir'l contraction}$$

only contracts curves tangent to $\mathcal{F}$

s.t. either

1) $K_{\mathcal{F}'}$ is nef $\quad \mathcal{F}' = \phi_* \mathcal{F}$

2) $\exists f : X \longrightarrow Z \quad -K_{\mathcal{F}}$ is f-ample and fibers of f are tangent to $\mathcal{F}$

<u>Remark</u> Can prove this for F-dlt pairs

Today Cone theorem

Thm (Cone + contraction theorem)

X proj. 3-fold $(\mathcal{F}, \Delta)$ be a log canonical pair.

$$\overline{NE}(X) = \overline{NE}(X)_{K_{\mathcal{F}}+\Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[\xi_i]$$

ξ_i is a rat'l curve tangent to $\mathcal{F}$

$$0 \leq -(K_{\mathcal{F}}+\Delta)\cdot \xi_i \leq 6$$

Moreover, if $(\mathcal{F}, \Delta)$ canonical then $\exists$ a contraction morphism associated to $\mathbb{R}_{\geq 0}[\xi_i]$ for all i.

π-exceptional divisor

$\exists$ canonical singularities provided
$a_i \geq 0 \quad \forall i, \pi.$

$\exists$ log canonical singularities provided
that $a_i \geq \begin{cases} 0 & \text{if } E_i \text{ is invariant} \\ -1 & \text{if } E_i \text{ is not invariant} \end{cases}$

Simple singularities (corank 1)

Let $\mathcal{F}$ be a corank 1 foliation on a germ $0 \in \mathbb{C}^n$

$\mathcal{F}$ has simple singularities provided formally about 0 $\mathcal{F}$ is generated by a 1-form of the following form

i) $\displaystyle\sum_{i=1}^{s} \lambda_i \frac{dx_i}{x_i}$ s.t. $\sum n_i \lambda_i = 0 \iff n_i = 0 \ \forall i$, $n_i \in \mathbb{Z}_{\geq 0}$

ii) $\displaystyle\sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \psi(x_1^{p_1} \cdots x_k^{p_k}) \sum_{i=2}^{s} \lambda_i \frac{dx_i}{x_i}$

s.t. $\sum n_i \lambda_i = 0 \iff n_i = 0 \ \forall i$

Remark $\{\prod_{i=1}^{s} x_i^{n_i} = 0\}$ $n_i \in \mathbb{Z}_{\geq 0}$ only invariant (formal) hypersurface

Warning! not all the x_i are convergent!

Remark $\dim \leq 3$ we can always resolve a foliation to one w/ simple sing (Seidenberg, Cano)

In higher dimensions ....?

Bend + Break (Keel-Matsuki-McKernan, Miyaoka).

X proj. variety, normal

$\mathcal{F}$ be a foliation

$D_1, \ldots, D_n$ $n = \dim X$ be nef $\mathbb{Q}$-Cartier divisors.

$D_1 \cdot \ldots \cdot D_n = 0$

$-K_{\mathcal{F}} \cdot D_2 \cdot \ldots \cdot D_n > 0$

Then, X is covered by rat'l curves Σ

- $-K_{\mathcal{F}} \cdot \Sigma > 0$
- tangent to $\mathcal{F}$
- $D_1 \cdot \Sigma = 0$.

Adjunction

Let $D \subseteq X$ be a divisor, $\mathcal{F}$ foliation
compute $(K_{\mathcal{F}} + \varepsilon(D) D)|_D$ in an "intelligent" way.

$$\varepsilon(D) = \begin{cases} 0 & \text{if } D \text{ is invariant} \\ 1 & \text{if } D \text{ is not invariant} \end{cases}$$

<u>$\varepsilon(D) = 0$</u>

By definition we have a morphism

$$T_{\mathcal{F}}|_D \longrightarrow T_D$$

$$(T_{\mathcal{F}}|_D)^{**} \longrightarrow T_D^{**} = T_D$$

giving an induced foliation on D, call it $\mathcal{F}_D$.

<u>Fact</u> $K_{\mathcal{F}}|_D = K_{\mathcal{F}_D} + \Delta_D \qquad \Delta_D \geq 0$

(Δ_D is supported on $\mathrm{sing}(\mathcal{F}) \cap D$)

<u>Remark</u> · special case where $\mathrm{rank}\, \mathcal{F} = 1$.

in this case $K_{\mathcal{F}_D} = K_D$.

· let S_i be any collection of $\mathcal{F}$ invariant divisors (possibly even formal!

$$(K_X + D + \sum S_i)|_D = K_D + \Theta_D$$

$\Theta_D \geq 0$ (classical adjunction formula)

<u>observe</u> $\Delta_D \geq \Theta_D$ (will be important later)

<u>$\varepsilon(D) = 1$</u>

$N^*_{\mathcal{F}} \subseteq \Omega^1_X$

$$\begin{tikzcd} N^*_{\mathcal{F}}|_D \arrow[r] \arrow[rr, bend left, "\psi"] & \Omega^1_X|_D \arrow[r] & \Omega^1_D \end{tikzcd}$$

$\mathrm{im}(\psi)$ defines a foliation on D.

call this foliation $\mathcal{F}_D$.

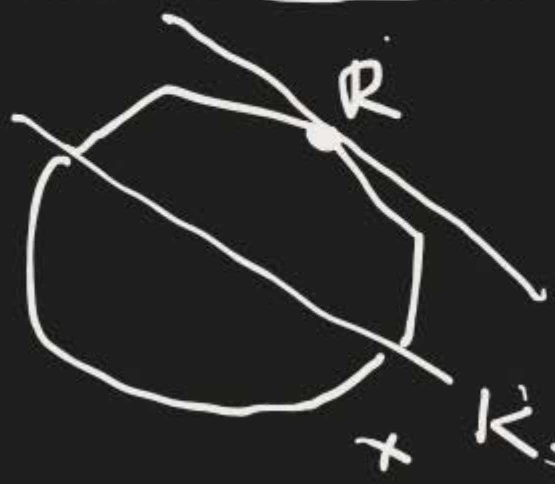
$(K_{\mathcal{F}} + D)|_D = K_{\mathcal{F}_D} + \Delta_D$ $\quad \Delta_D \geq 0$ is canonically defined divisor.

Remark.

- $(\mathcal{F},\Delta)$ log canonical then $(\mathcal{F}_\Delta, \Delta_\Delta)$ is log canonical.

- $\mathrm{rk}\,\mathcal{F}=1$ $(\mathcal{F},D)$ is log canonical
  $(K_\mathcal{F}+D)|_D \sim 0$
  ($K_{\mathcal{F}_D}=0$ for the foliation by points).

Proof of cone theorem $\Delta=0$



$H_R = K_\mathcal{F}+A$ $\quad$ A ample

$\times\, K_\mathcal{F}$ $\quad$ nef

goal show R is spanned by a rat'l

curve.

(i) $H_R^3 = 0$ (not big)

(ii) $H_R^3 > 0$ (big)

(i) apply bend & break

$D_i = H_R \qquad i \leq \nu(H_R)$

$D_i = A \qquad 3 \geq i > \nu(H_R)$

produces rat'l curve Σ s.t.

$\Sigma \cdot H_R = 0 \implies [\Sigma] \in R$. ✓

(ii) proceed by induction on dimension

$H_R \sim_{\mathbb{Q}} G + E \qquad G$ ample $\qquad E \geq 0$

In particular $\exists\ E_0 \subseteq \mathrm{Supp}(E)$ $E_0 \cdot R < 0$.

$R \subseteq \mathrm{im}(\overline{NE}(E_0) \longrightarrow \overline{NE}(X))$

Want to restrict to E_0 and apply the cone theorem in $\dim = 2$

(Bogomolov - McQuillan)

(i) E_0 is invariant

(ii) E_0 is not invariant

Focus (ii) ((i) is similar)

$(K_{\mathcal{F}} + E_0) \cdot R < 0$

$(K_{\mathcal{F}} + E_0)|_{E_0} = K_{\mathcal{F}_{E_0}} + \Delta_{E_0}$ by adjunction

$\exists\ R_0 \subseteq \overline{NE}(E_0)$ s.t. R_0 is extremal and $(K_{\mathcal{F}_{E_0}} + \Delta_0)$-negative and s.t. R_0 maps to R under the inclusion.

Apply the cone theorem for surface foliation to see that R_0 is spanned by a rat'l curve and so R . $\square$