Existence of contractions + flips

X 3-fold $\exists$ canonical

$R \subseteq \overline{NE}(X)$ be $-$ K_X-negative extremal ray

$$\mathrm{loc}(R) = \{ P \in C \mid [C] \in R \}.$$

Fact $\mathrm{loc}(R)$ is a subvariety

$\mathrm{loc}(R) = X$

In this case R is K_X-negative (not hard to show)

can realize the K_X contraction as

a K_X contraction

$\underline{\text{Exc}(R) = D \text{ is a divisor}}$

D is not invariant .... leave as an exercise

D is normal

$$K_Y|_D = K_D + \Delta_D$$

$$(K_X + D) = K_D + \Theta_D$$

<u>Recall</u> $\Theta_D \leq \Delta_D$

this implies that R is $K_X + D$ negative.
need to control the singularities of the pair (X, D).

<u>observation</u> if $\mathcal{F}$ has canonical singularities then (X, D) has log canonical singularities.

In particular, the $K_X + D$-negative contraction exists.

<u>Remark</u> gives a hint that we should construct $K_{\mathcal{F}}$ flips by constructing K_X log flips.

<u>loc(R)</u> $= C$ a curve (flipping case)

More singularities

<u>Def</u> X be a variety $\mathcal{F}$ a codim 1 foliation

we say $\mathcal{F}$ is $\mathcal{F}$-dlt if $\exists \pi : \bar{X} \longrightarrow X$

s.t. i) $\bar{\mathcal{F}} := \pi^{-1}\mathcal{F}$ has simple singularities

(in particular $\bar{X}$ is smooth)

ii) if we write $K_{\bar{\mathcal{F}}} = \pi^* K_{\mathcal{F}} + \sum a_i E_i$

$$E_i > -\varepsilon(E_i)$$

<u>Remark</u> - "next best thing to simple sing"
- foliated analogue of dlt sing
- version for pairs $(\mathcal{F}, \Delta)$
- $\mathcal{F}$ dlt then X is klt.
- $\mathcal{F}$ canonical then X is klt

<u>WARNING</u> Canonical $\not\Rightarrow$ F-dlt

e.g. $x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$

- Running the MMP ~~model~~ starting w/ simple sing. produces ~~vers~~ foliation w/ F-dlt sing.

- Let Z be a codim $=2$ component of $\mathrm{sing}(\mathcal{F})$, then Z is contained in 2 (formally) invariant divisors.*

<u>Thm</u> F-dlt flips exist.

Set up
$$\begin{tikzcd} X \arrow[d, "f"'] & \supseteq & C \arrow[d] \\ Z & \ni & p \end{tikzcd}$$
$\leftarrow$ flipping curve.

C might have multiple components, but each component is tangent to $\mathcal{F}$.

Let S_1 be the collection of all

invariant divisor meeting C.

Fact (Cano–Cerveau) if $\hat{X}$ is the formal completion of X along C then we can extend S_i as a formal subvariety of $\hat{X}$.

(⊛) **Fact** $\left(K_{\hat{\mathcal{F}}} - (K_{\hat{X}} + \sum S_i)\right)\big|_{\hat{X}}$ is nef.

(This requires F-dlt singularities)

Pf restrict the strong sep.

$\Delta_D \geq \Theta_D$ w/ equality at the generic point of C ∎

In particular $(K_{\hat{X}} + \varepsilon S_{\eta}).C < 0$

Problems i) want to produce the log flip but we don't the MMP in the formal setting

technical $\left\{ \begin{array}{l} \text{ii) } Z \text{ algebraic space... need to prove that it's projective} \\ \text{iii) } \hat{X} \text{ ~~need~~ is not necessarily } \mathbb{Q}\text{-factorial} \end{array} \right.$

How can we produce a "formal" flip?

Remark producing the flip is étale local on Z, so we can freely replace Z by an étale nbhd of P. *

Technical ingredient

Thm (Corollary of Artin/Elkik approx.)
(A, m) henselian local ring $\hat{A}$ be the completion.
Let $\hat{M}$ be a $\hat{A}$-module and
$\hat{s} \in H^0(\operatorname{Spec} \hat{A}, \hat{M})$.

Suppose $\hat{M}$ is locally free on $\operatorname{Spec}\hat{A} \setminus \{m\}$. Then for all $n > 0$

$\exists$ A-module M and $s \in H^0(\operatorname{Spec} A, M)$ s.t. $M \otimes \hat{A} \cong \hat{M}$ $s = \hat{s}$ mod m^n. $\square$

How do we use this?

$\hat{A} = \widehat{\mathcal{O}_{Z,P}}$ $\quad \hat{X} \xrightarrow{f} \operatorname{Spec}\hat{A}$

$\hat{M} = \mathcal{O}(f_* S_i)$ reflexive sheaf

- S_i is $\mathbb{Q}$-Cartier $\Rightarrow$ $\mathcal{O}(f_* S_i)$ is $\mathbb{Q}$-Cartier away from P

Cheating a bit we can assume $\mathcal{O}(f_* S_i)$ is locally free away from P.

$A =$ henselization of $\mathcal{O}_{Z,p}$

$\Rightarrow \exists$ M on $\operatorname{Spec} A$ s.t. $M \otimes \hat{A} = \mathcal{O}(f_* S_i)$

aka $\exists$ $s \in H^0(\operatorname{Spec} A, M)$ s.t. s approximates the tautological section of $\mathcal{O}(f_* S_i)$.

Henselizations are the local rings for the étale topology

so $\exists$ $Z' \longrightarrow Z$ étale and a reflexive sheaf M' on Z' and a section s' which give (M, s) in $\operatorname{Spec} A$.

$\tilde{S}_i := \{ s' = 0 \}$

$\tilde{S}_i = S_i$ mod m^n for some $n \gg 0$

$X' = X \times_Z Z'$

can construct the $K_{X'} + \sum \tilde{S}_i$ flip.

Can check for all $n \gg 0$ the flip is independent of all our choices of approximation and so gives the K_Z - flip $\blacksquare$

<u>Termination</u> · show flips become disjoint from lc centre
· Bott's partial connection to show that terminal $K_{\mathcal{F}}$ flips are K_X-flips.

<u>Fact</u> Basepoint free theorem is false for foliations, and so it isn't clear how to construct canonical models.

<u>Thm</u> $(\mathcal{F}, \Delta)$ is an F-dlt pair $\Delta = A + B$
A ample $B \geq 0$ (e.g. $\mathcal{F}$ simple sing, $\Delta < A$ is ample).
$K_{\mathcal{F}} + \Delta$ is nef.
then $K_{\mathcal{F}} + \Delta$ is semi-ample.

Pf follows from the conext contraction on non-$\mathbb{Q}$-factorial varieties ■

+ canonical bundle formula
for fibrations.