PROBLEMS TO THINK ABOUT

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Contents

1. MMP for toric foliations

Let X be a proper toric variety and let $\mathcal{F}$ be a foliation on X such that $\mathcal{F}$ is invariant under the action of the torus. For example, on $\mathbb{A}^2$ with coordinates x, y $\omega = \frac{dx}{x} + \lambda\frac{dy}{y}$ defines a torus invariant foliation. In [Spi] it was shown that the MMP can be run for co-rank 1 torus invariant foliations in all dimensions.

Question 1. *Can we run the MMP for torus invariant foliations of any rank?*

2. MMP with a group action

It is known that if we have a variety X together with an action of a finite group G on X then it is possible to run a G-equivariant MMP.

Suppose that X is a threefold and $\mathcal{F}$ is a foliation on X and G is a finite group acting on X which leaves $\mathcal{F}$ invariant.

Question 2. *Can we run a G-equivariant foliated MMP?*

3. Foliations on surface singularities

Let $P \in X$ be a germ of a (possibly singular) surface together with a rank 1 foliation $K_{\mathcal{F}}$. Suppose that $\mathcal{F}$ has a canonical singularity at P. It is known by [McQ08] that $P \in X$ is in fact a quotient singularity.

However, as examples show, if $\mathcal{F}$ is log canonical at P there is in general no such "bound" on the singularities of the underlying space. It is a good exercise to try and write down some examples of your own!

Question 3. *Suppose $P \in X$ is a rational surface singularity and that $\mathcal{F}$ is log canonical at P. Is $P \in X$ a quotient singularity?*

4. Complements for Fano foliations

The theory of complements has proven to be very important in the study of Fano varieties. It would be interesting to understand complements for Fano foliations.

Let X be a normal projective variety and let $\mathcal{F}$ be a log canonical foliation. By a complement we mean a divisor $0 \leq D \in |-K_{\mathcal{F}}|$ such that $(\mathcal{F}, D)$ is log canonical.

Question 4. *Do foliated complements always exist?*

Following the work of Araujo and Druel, see for example, [AD13], or the videos in their mini-course, it might make sense to approach this question first for foliations of high index on smooth varieties.

It also makes sense to ask this question in the relative situation, i.e., let $\pi \colon X \to U$ be a proper morphism and such $-K_{\mathcal{F}}$ is π-ample. Do foliation complements exist in this case? This seems to be a harder problem, and some cases of this are used in our approach to the rank 1 MMP.

5. Classifying log canonical singularities

This question can be viewed as a local version of the existence of complements for rank 1 foliations.

Let $P \in X$ be a germ of a variety, let $\mathcal{F}$ be a rank 1 foliation on X and let $0 \leq D$ be a $\mathbb{Q}$-divisor such that $K_{\mathcal{F}} + D$ is $\mathbb{Q}$-Cartier and $(\mathcal{F}, D)$ is log canonical.

Question 5. *Does there exist a $\mathbb{Z}$-divisor D' such that $K_{\mathcal{F}} + D'$ is $\mathbb{Q}$-Cartier and $(\mathcal{F}, D')$ is log canonical?*

This question is a good exercise when $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier itself, but if $K_{\mathcal{F}}$ is not $\mathbb{Q}$-Cartier this problem seems much more challenging. Having an affirmative answer to this question would help provide a strong classification result for rank 1 log canonical foliation singularities.

6. Automorphisms of foliations

It is a classical result for a curve C with $g(C) \geq 2$ that $\mathrm{Aut}(C)$ is finite, and in fact is bounded by $84(g(C)-1)$. This was later generalized (using techniques from the MMP) to a statement for varieties of general type (i.e., with K_X-big) in all dimensions, [HMX13].

Question 6. *Let X be a smooth projective variety of dimension ≤ 3 and let $\mathcal{F}$ be a foliation of general type with $K_{\mathcal{F}}$ nef. Can we find a bound on $Aut(\mathcal{F})$ in terms of $K_{\mathcal{F}}^{dim(X)}$?*

See [CF14] for some partial results in this direction.

This question is related to the following question.

Question 7. *Let X be a projective surface (not necessarily smooth!) and let $\mathcal{F}$ be a foliation with canonical singularities and $K_{\mathcal{F}}$ nef. Can one produce a lower bound for $K_{\mathcal{F}}^2$ independent of $\mathcal{F}$?*

We direct you to [HL] (available on arXiv) for some considerations around this problem.

References

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