FANO FOLIATIONS - LIST OF PROBLEMS

CAROLINA ARAUJO AND STÉPHANE DRUEL

Contents

1. Fano foliations with large index 1
2. Boundedness for Fano foliations 2
3. Geometry of webs on projective manifolds 2
References 2

1. Fano foliations with large index

1.1. Del Pezzo foliations. The papers [AD13], [AD16] and [Fig19] give a classification of algebraically integrable del Pezzo foliations of rank $\geqslant 3$ with log canonical singularities (in the sense of [AD13, Definition 3.7]). The following is still missing.

Question 1.1. Classify algebraically integrable del Pezzo foliations of rank 2 with log canonical singularities.

The following is the classification of all possible general log leaves of del Pezzo foliations on smooth projective varieties.

Proposition 1.2 ([Ara16])**.** *Let $\mathscr{F}$ be an algebraically integrable del Pezzo foliation of rank $r \geqslant 2$ on a smooth projective variety X, with general log leaf (F, Δ). Let L be an ample divisor on X such that $-K_{\mathscr{F}} \sim (r-1)L$. Then $\big(F, \Delta, L_{|F}\big)$ satisfies one of the following conditions.*

(1) $\big(F, \mathscr{O}_F(\Delta), \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}^r, \mathscr{O}_{\mathbb{P}^r}(2), \mathscr{O}_{\mathbb{P}^r}(1))$.
(2) *(F, Δ) is a cone over (Q^m, H), where Q^m is a smooth quadric hypersurface in $\mathbb{P}^{m+1}$ for some $2 \leqslant m \leqslant r$, $H \in \big|\mathscr{O}_{Q^m}(1)\big|$, and $L_{|F}$ is a hyperplane under this embedding.*
(3) $\big(F, \mathscr{O}_F(\Delta), \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1), \mathscr{O}_{\mathbb{P}^2}(2))$.
(4) *$\big(F, \mathscr{O}_F(L_{|F})\big) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$, and one of the following holds:*
   (a) *$\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ for some $d \geqslant 2$, and $\Delta \sim_{\mathbb{Z}} \sigma + f$, where σ is the minimal section and f a fiber of $\mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$.*
   (b) *$\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ for some $d \geqslant 2$, and Δ is a minimal section.*
   (c) *$\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$ for some $d \geqslant 1$, and $\Delta = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1))$.*
(5) *(F, Δ) is a cone over (C_d, B), where C_d is rational normal curve of degree d in $\mathbb{P}^d$ for some $d \geqslant 2$, $B \in \big|\mathscr{O}_{\mathbb{P}^1}(2)\big|$, and $L_{|F}$ is a hyperplane under this embedding.*
(6) *(F, Δ) is a cone over the pair (4a) above, and $L_{|F}$ is a hyperplane section of the cone.*

Question 1.3. Which possible general log leaves of del Pezzo foliations do actually occur for del Pezzo foliations on smooth projective varieties?

See [Ara16, Remark 2.10] for some examples of non-log canonical general log leaves of del Pezzo foliations.

The structure of del Pezzo foliations on $\mathbb{P}^m$-bundle $\pi : X \to \mathbb{P}^l$ is given in [AD13]. Recall from [AD13, Theorem 1.1] that $\mathscr{F}$ is then algebraically integrable with rationally connected general leaves.

If $m = 1$, then $X \simeq \mathbb{P}^1 \times \mathbb{P}^l$, and $\mathscr{F}$ is the pullback via π of a foliation $\mathscr{O}(1)^{\oplus i} \subset T_{\mathbb{P}^l}$ for some $i \in \{1, 2\}$.

Theorem 1.4 ([AD13, Theorem 1.4])**.** *Let $\mathscr{F} \subsetneq T_X$ be a del Pezzo foliation on a $\mathbb{P}^m$-bundle $\pi : X \to \mathbb{P}^l$, with $m \geq 2$. Suppose that $\mathscr{F} \not\subset T_{X/\mathbb{P}^l}$. Then there is an exact sequence of vector bundles $0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$ on $\mathbb{P}^l$ such that $X \simeq \mathbb{P}_{\mathbb{P}^l}(\mathscr{E})$, and $\mathscr{F}$ is the pullback via the relative linear projection $X \dashrightarrow Z = \mathbb{P}_{\mathbb{P}^l}(\mathscr{K})$ of a foliation $q^* \det(\mathscr{Q}) \subset T_Z$. Here $q : Z \to \mathbb{P}^l$ denotes the natural projection. Moreover, one of the following holds.*

(1) *$l = 1$, $\mathscr{Q} \simeq \mathscr{O}(1)$, $\mathscr{K}$ is an ample vector bundle such that $\mathscr{K} \not\simeq \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}$ for any integer a, and $\mathscr{E} \simeq \mathscr{Q} \oplus \mathscr{K}$ $(r_{\mathscr{F}} = 2)$.*
(2) *$l = 1$, $\mathscr{Q} \simeq \mathscr{O}(2)$, $\mathscr{K} \simeq \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus m}$ for some integer $a \geqslant 1$, and $\mathscr{E} \simeq \mathscr{Q} \oplus \mathscr{K}$ $(r_{\mathscr{F}} = 2)$.*
(3) *$l = 1$, $\mathscr{Q} \simeq \mathscr{O}(1) \oplus \mathscr{O}(1)$, $\mathscr{K} \simeq \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus(m-1)}$ for some integer $a \geqslant 1$, and $\mathscr{E} \simeq \mathscr{Q} \oplus \mathscr{K}$ $(r_{\mathscr{F}} = 3)$.*
(4) *$l \geqslant 2$, $\mathscr{Q} \simeq \mathscr{O}(1)$, and $\mathscr{K}$ is V-equivariant for some $V \in H^0\big(\mathbb{P}^l, T_{\mathbb{P}^l} \otimes \mathscr{O}(-1)\big) \setminus \{0\}$ $(r_{\mathscr{F}} = 2)$.*

Conversely, given $\mathscr{K}$, $\mathscr{E}$ and $\mathscr{Q}$ satisfying any of the conditions above, there exists a del Pezzo foliation of that type.

Question 1.5. Let $\mathscr{F}$ be a above. Describe the general log leaf of $\mathscr{F}$. Then deduce those foliations with log canonical singularities in the sense of [AD13, Definition 3.7].

Question 1.6. Describe foliations as above with log canonical singularities in the sense of McQuillan.

1.2. Mukai foliations. Let $\mathscr{F}$ be a Mukai foliation of rank r on an n-dimensional complex projective manifold $X \not\cong \mathbb{P}^n$ (i.e., the index of $\mathscr{F}$ satisfies $\iota_{\mathscr{F}} = r - 2 \geqslant 1$). We know from [AD19, Corollary 1.6] that the algebraic rank of $\mathscr{F}$ satisfies $r^a_{\mathscr{F}} \geqslant \iota_{\mathscr{F}} + 1$. In [AD17], we have classified codimension 1 Mukai foliations on projective manifolds. It follows from this classification that, in the codimension 1 case, if $r^a_{\mathscr{F}} = \iota_{\mathscr{F}} + 1$, then

(1) either $X \cong Q^n \subset \mathbb{P}^{n+1}$, and $\mathscr{F}$ is the pullback under the restriction to X of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^2$ of a foliation on $\mathbb{P}^2$ induced by a global vector field;
(2) or X is a projective space bundle over a curve or a surface, and $\mathscr{F}$ is the pullback of a codimension 1 foliation on a surface or threefold.

Question 1.7. Describe Mukai foliations $\mathscr{F}$ of arbitrary rank r for which $r^a_{\mathscr{F}} = \iota_{\mathscr{F}} + 1 = r - 1$. One may start with the case of codimension 2.

2. Boundedness for Fano foliations

For Fano varieties, we have the following boundedness results.

Theorem 2.1 ([KMM92])**.** *For fixed n, Fano manifolds of dimension n form a bounded family.*

If we allow Fano varieties with arbitrary singularities, then boundedness fails, already in dimension 2. Consider for instance cones over rational normal curves of degre $n > 0$. They all have *klt* singularities, and clearly do not form a bounded family. On the other hand, if one suitably bounds the singularities, then boundedness still holds for singular Fano varieties of fixed dimension. More precisely:

Theorem 2.2 ([Bir16])**.** *For fixed positive integer n and positive real number ε, Fano varieties of dimension n with ε-log canonical singularities form a bounded family.*

Question 2.3. For fixed positive integers r and n, do Fano foliations $\mathscr{F}$ of rank r on projective manifolds of dimension n form a bounded family, possibly imposing restrictions on the singularities of $\mathscr{F}$?

3. Geometry of webs on projective manifolds

Study global properties of webs (see [PP15] for this notion) following ideas from the minimal model program.

Question 3.1. Define the canonical class of a web.

Question 3.2. Is there a Kobayashi-Ochiai theorem for Fano webs?

Question 3.3. Define notions of singularities for webs.

Question 3.4. Study the geometry of Fano webs with large index (and mild singularities).

References

[AD13] Carolina Araujo and Stéphane Druel, *On Fano foliations*, Adv. Math. **238** (2013), 70–118.

[AD16] ———, *On Fano foliations 2*, Foliation theory in algebraic geometry. Proceedings of the conference, New York, NY, USA, September 3–7, 2013, Cham: Springer, 2016, pp. 1–20.

[AD17] ———, *Codimension 1 Mukai foliations on complex projective manifolds*, J. Reine Angew. Math. **727** (2017), 191–246.

[AD19] Carolina Araujo and Stéphane Druel, *Characterization of generic projective space bundles and algebraicity of foliations*, Comment. Math. Helv. **94** (2019), no. 4, 833–853.

[Ara16] Carolina Araujo, *Positivity and algebraic integrability of holomorphic foliations*, Foliation theory in algebraic geometry. Proceedings of the conference, New York, NY, USA, September 3–7, 2013, Cham: Springer, 2016, pp. 1–20.

[Bir16] Caucher Birkar, *Singularities of linear systems and boundedness of fano varieties*, Preprint `arXiv:1609.05543`, 2016.

[Fig19] João Paulo Figueredo, *Del pezzo foliations with log canonical singularities*, Preprint arXiv:1909.03401, 2019.

[KMM92] J. Kollár, Y. Miyaoka, and S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), no. 3, 765–779.

[PP15] Jorge Vitório Pereira and Luc Pirio, *An invitation to web geometry*, IMPA Monographs, vol. 2, Springer, Cham, 2015. MR 3309231