Part 3: Greedy energy minimization and the van der Corput sequence

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Steinerberger recently proposed to study whether regular sequences could be constructed via dynamical systems.

Suppose we are given $\{x_0, \ldots, x_{N-1}\} \subset [0, 1)$, then he proposed to construct x_N in a **greedy manner** as

$$x_N = \arg\min \sum_{k=0}^{N-1} 1 - \log(2 \sin(\pi |x - x_k|)),$$

and if the minimum is not unique, any choice is admissible.

Greedy algorithm II

Steinerberger proves that, independently of the initial conditions, such sequences satisfy $D_N \leq cN^{-1/2} \log N$. Moreover, he **conjectures**:

(A) (**weak form**): the discrepancy of every such sequence satisfies

$$D_N \leq cN^{-1} \log N;$$

(B) (**strong form**): and the constant c does not depend on the initial set from which this construction is started.

We give **affirmative answers** to these conjectures in the **special case** when we are given only one initial point x_0.

Let $a_j(n)$ denote the jth coefficient in the binary representation

$$n = a_0 \cdot 2^0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \ldots = \sum_{j=1}^{\infty} a_j 2^j$$

of an integer n, in which $a_j(n) \in \{0, 1\}$ and if $0 \leq n < 2^m$, then $a_j(n) = 0$ for all $j \geq m$. The binary radical inverse function is defined as $S_2 : \mathbb{N}_0 \to [0, 1)$,

$$S_2(n) = \frac{a_0(n)}{2} + \frac{a_1(n)}{2^2} + \frac{a_2(n)}{2^3} + \ldots = \sum_{j=0}^{\infty} \frac{a_j(n)}{2^{j+1}}.$$

Then the classical van der Corput sequence is defined as $\mathcal{S}_2 = (S_2(n))_{n \geq 0}$.

Permuted van der Corput sequences II

Henri Faure (Luminy!!) generalised the definition in two ways.

He replaced the binary representation of an integer by its general b-adic representation for a fixed integer base $b \geq 2 \Rightarrow$ the **b-adic radical inverse function** S_b, which is used to define vdC sequences in general bases; i.e. $\mathcal{S}_b = (S_b(n))_{n\geq 0}$.

Furthermore, the **generalised (or permuted) van der Corput sequence** $\mathcal{S}_b^\sigma = (S_b^\sigma(n))_{n\geq 0}$ for a fixed base $b \geq 0$ is defined via the permuted b-adic radical inverse function

$$S_b^\sigma(n) = \sum_{j=0}^{\infty} \frac{\sigma(a_j(n))}{b^{j+1}}.$$

Every such sequence is uniformly distributed modulo 1.

A Family of permutations I

We **inductively** define a set of permutations $\mathcal{P}_m \subset \mathfrak{S}_{2^m}$ in each basis $b_m = 2^m$.

We start with $b_1 = 2$ and $\mathcal{P}_1 = \{(0,1)\}$ and we obtain the set $\mathcal{P}_{m+1}$ from $\mathcal{P}_m$ in the following way:

We first multiply each permutation $\sigma \in \mathcal{P}_m$ with 2 and denote the resulting tuple of numbers as 2σ and the set of all such tuples as $2\mathcal{P}_m$.

A family of permutations II

Next, each $2\sigma \in 2\mathcal{P}_m$ gives rise to 2^m new tuples:

For each odd a with $1 \leq a \leq 2^{m+1}$ we form a new tuple $2\sigma \oplus a$ by adding a to 2σ (addition modulo 2^{m+1}). The set of all such tuples is denoted by $2\mathcal{P}_m \oplus a$.

Finally, the set $\mathcal{P}_{m+1}$ is defined as the set of all permutations $(2\sigma, 2\sigma' \oplus a)$ for $\sigma, \sigma' \in \mathcal{P}_m$ and odd a with $1 \leq a \leq 2^{m+1}$.

A family of permutations III

As examples we construct $\mathcal{P}_2$ and $\mathcal{P}_3$.

First, we have that $2\mathcal{P}_1 = \{(0,2)\}$ and $2\mathcal{P}_1 \oplus a = \{(1,3),(3,1)\}$.

Consequently $\mathcal{P}_2 = \{(0,2,1,3),(0,2,3,1)\}$. Next, we have

$$2\mathcal{P}_2 = \{(0,4,2,6),(0,4,6,2)\},$$

and

$$\begin{aligned} 2\mathcal{P}_2 \oplus a = \{&(1,5,3,7),(3,7,5,1),(5,1,7,3),(7,3,1,5),\\ &(1,5,7,3),(3,7,1,5),(5,1,3,7),(7,3,5,1)\} \end{aligned}$$

from which we can build $\mathcal{P}_3$.

Main theorem

Theorem (P. (2019))

Let $f : [0,1] \to \mathbb{R}$, *be a function satisfying*

(i) $f(x) = f(1-x)$;

(ii) f *is twice differentiable on* $(0,1)$;

(iii) $f''(x) > 0$ *for all* $x \in (0,1)$;

and let the sequence $X = (x_N)_{N=0}^{\infty} \subset [0,1)$ *be defined by* $x_0 = 0$ *and*

$$x_N = \arg\min_{x\in(0,1)} \sum_{k=0}^{N-1} f(|x - x_k|),$$

for $N > 0$ *and where every global minimum is admissible if it happens not to be unique.*

Then *for every fixed* $N > 0$ **there exist** *an* $m > 0$ *with* $N \leq 2^m$ *and a* $\sigma \in \mathcal{P}_m$ **such that** $x_k = S_{2^m}^{\sigma}(k)$ *for all* $1 \leq k \leq N$ *and* $D_N(X) = D_N(\mathcal{S}_{2^m}^{\sigma}) = D_N(\mathcal{S}_2)$.

Remarks I

We believe the Theorem is a strong indicator that the dynamical version

$$x_N = \arg\min_x \sum_{k=0}^{N-1} f(|x - x_k|),$$

of the static equilibrium problem in mathematical physics might give rise to **interesting structures**.

In the one-dimensional case, it certainly connects in a very substantial way to structures in number theory. To emphasize this, we explicitly state that:

> **Greedy energy minimization on the one-dimensional torus automatically recovers the way we count in binary.**

Remarks II

Question

Which structures arise when we use different classes of function that are not covered by the theorem? Can we get similar structures? Or totally different dynamics?

Remarks III

The result has nontrivial implications for the study of uniform distribution.

It starts by providing a **novel definition** of the van der Corput sequence; i.e. start the greedy algorithm with $\{0\}$ and always pick the smallest of the suggested minima.

The second statement in the main Theorem, i.e. Discrepancy being preserved over all possible choices, shows that potential theoretic approaches along the lines of what was proposed by Steinerberger might indeed have **intimate ties** to discrepancy.

Question

Is there another family of functions which can be used to reconstruct permuted van der Corput sequences in prime bases $b > 2$ via greedy energy minimisation?

Question

We have verified Steinerberger's conjectures for the case of one initial element. What about the case of an arbitrary set $\{x_0, x_1\}$ of two (or more) points in $[0, 1]$?

We suspect that the classical van der Corput sequence and its permutations form in a way a unique (and natural) link between sequences constructed via greedy energy minimisation and sequences obtained from traditional methods in number theory.

References

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Thank you