Eigenvalue Distributions of Wigner and Wishart Ensembles of Vinberg Matrices

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Abstract

This is a first step toward studying eigenvalue distributions of Wigner and Wishart Ensembles on matrix spaces related to growing graphical models (see [1]). Growing daisy graphs are among the most natural classes of such graphical models. Vinberg matrices are the symmetric matrices corresponding to the growing daisy graphs. Since the space of Vinberg matrices is endowed with transitive group actions, covariance matrices are defined naturally. We provide a complete study of limiting eigenvalue distributions related to Vinberg matrices; For Winger ensembles, we give an explicit formula of the limiting eigenvalue distributions, and for Wishart ensembles, their limiting Stieltjes transforms are described explicitly in terms of the generalized Lambert functions.

Theorem 1 (Wigner Ensembles)

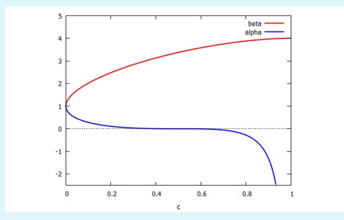
Let U_n be a Wigner matrix in $\mathcal{U}_n$ with variance 1. Assume that $\lim_{n\to+\infty} \frac{a_n}{n} = c$. Then, the limiting eigenvalue distributions μ of $\frac{1}{\sqrt{n}}U_n$ exists and is given as

$$\mu(t) = \frac{\sqrt[3]{R_c^+(t)} - \sqrt[3]{R_c^-(t)}}{2\sqrt{3}\,\pi\, t}\chi_{[\alpha_c,\,\beta_c]}(t^2) + [\,1-2c\,]_+\delta_0(t),$$

where $\delta_0(t)$ is the Dirac delta function at $t=0$, $[\,a\,]_+ := \max(a,0)$ for $a \in \mathbb{R}$ and

$$R_c^{\pm}(x) := x^6 - 3(c+1)x^4 + \frac{3}{2}(5c^2-2c+2)x^2 + (2c-1)^3 \pm 3c\sqrt{3-3c}\cdot x\sqrt{(x^2-\alpha_c)(\beta_c-x^2)},$$

$$\alpha_c = \frac{8+4c-13c^2-\sqrt{c(8-7c)^3}}{8\,(1-c)}, \quad \beta_c = \frac{8+4c-13c^2+\sqrt{c(8-7c)^3}}{8\,(1-c)}.$$



Sketch of the proofs

For the proofs, we use the Variance profile method (cf. [2, Theorem3.1], [4]).

Variance profiles

A variance profile $\sigma\colon [0,1]\times[0,1] \to \mathbb{R}_{\geq 0}$ is roughly speaking a matrix having information of variances of random matrices; e.g. if σ is continuous, then the variance of $\sqrt{n}Y_{ij}$ is roughly $\sigma(\frac{i}{n},\frac{j}{n})^2$.

Variance profile methods

Let Y_{ij} $(i \leq j)$ be i.i.d. centered real variables with suitable moment conditions (moderate 2nd, 4th moments). Assume that $Y = (Y_{ij})$ has a variance profile σ. Then, there exists the limiting eigenvalue distribution μ_σ, and its Stieltjes transform can be obtained by solving a suitable functional equation or integral equation.

The proof of Theorem 1

It reduces to solve the cubic equation $zX^3 - \left(z^2+2c-1\right)X^2 + 2czX - 1 = 0$.

The proof of Theorem 2

It reduces to solve the following simultaneous differential equation with suitable initial data:

$$a'(t) = -\alpha a(t)^2 b(t), \qquad b'(t) = a(t)b(t)^2 \qquad \left(0 \leq t \leq \frac{1}{\beta+1}\right).$$

Terminologies

Wigner matrices

A Vinberg matrix $U_n = (u_{ij}) \in \mathcal{U}_n$ is called a Wigner matrix with variance 1 if each u_{ij} is i.i.d., 0 mean and 1 variance whenever $u_{ij} \neq 0$.

Wishart matrices

A Vinberg matrix $X_n \in \mathcal{U}_n$ is called a Wishart matrix with respect to a subspace E of $\mathrm{Mat}(n\times N;\mathbb{R})$ if there exists an i.i.d. matrix $\eta_n = (\eta_{ij}) \in E$ such that $X_n = \eta_n{}^t\eta_n \in \mathcal{U}_n$ and η_{ij} has 0 mean and 1 variance whenever $\eta_{ij} \neq 0$.

Generalized Lambert functions

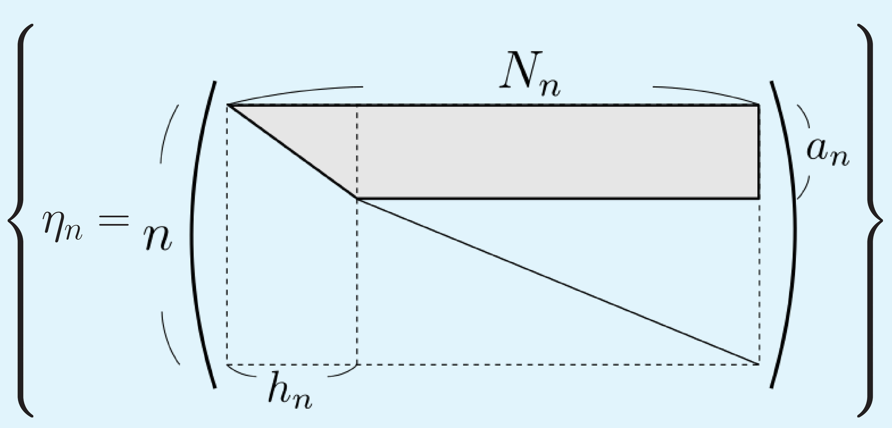
For $\kappa, \gamma \in \mathbb{R}$ such that $\frac{1}{\kappa} - \gamma > 0$, a generalized Lambert function $W_{\kappa,\gamma}$ is the main branch of the inverse function of a function $f_{\kappa,\gamma}$ defined by

$$f_{\kappa,\gamma}(x) := \frac{x}{1+\gamma x}\left(1+\frac{x}{\kappa}\right)^{\kappa} \quad \text{where} \quad 1+\frac{x}{\kappa} > 0.$$

Quadratic construction

A subspace $E_n = E_n(h_n, N_n)$ of $\mathrm{Mat}(n\times N_n, \mathbb{R})$ is said to be obtained by a quadratic construction if, roughly speaking, E_n is a subspace of $\mathrm{Mat}(n\times N_n;\mathbb{R})$ of the form

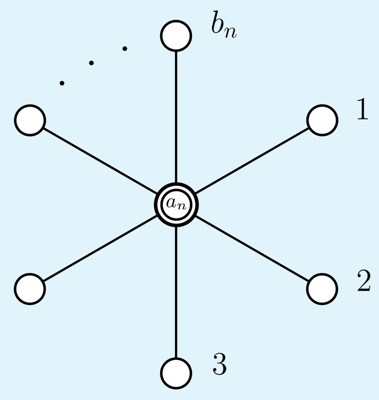
$$E_n(h_n, N_n) := \left\{ \eta_n = n \right.$$



For any $\eta_n \in E_n$, the matrix $X_n = \eta_n{}^t\eta_n$ is a Vinberg matrix, that is, $X_n \in \mathcal{U}_n$. Note that it appears naturally from the theory of homogeneous cones.

Daisy graphs and Vinberg Matrices

We want to know eigenvalue distributions related the following matrix spaces $\mathcal{U}_n$, the space of **Vinberg matrices**. Below, the double circle around the vertex a_n indicates the complete graph with a_n vertices.



$$\longleftrightarrow \quad \mathcal{U}_n = \left\{ U_n = \begin{pmatrix} x_{11} & \cdots & x_{1a_n} & y_{11} & \cdots & \cdots & y_{1b_n} \\ \vdots & \ddots & \vdots & \vdots & & & \vdots \\ x_{1a_n} & \cdots & x_{a_na_n} & y_{a_n1} & \cdots & \cdots & y_{a_nb_n} \\ y_{11} & \cdots & y_{a_n1} & d_1 & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & d_2 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ y_{1b_n} & \cdots & y_{a_nb_n} & 0 & \cdots & 0 & d_{b_n} \end{pmatrix} ; \begin{array}{l} x_{ij}\in\mathbb{R}, \\ y_{ij}\in\mathbb{R}, \\ d_j\in\mathbb{R} \end{array} \right\}$$

Theorem 2 (Wishart Ensembles)

Let $E_n = E_n(h_n, N_n)$ be a subspace of $\mathrm{Mat}(n\times N_n;\mathbb{R})$ obtained by a quadratic construction, and let X_n be a Wishart matrix with respect to E_n. Assume that

$$\lim_{n\to+\infty}\frac{a_n}{n} = c, \quad \lim_{n\to\infty}\frac{h_n}{a_n} = \alpha, \quad \lim_{n\to\infty}\frac{N_n}{a_n} = \beta.$$

Then, the limiting eigenvalue distributions of $\frac{1}{n}X_n$ exists, and its Stieltjes transform $S(z) = \int_{\mathbb{R}}\frac{\mu(dt)}{t-z}$ is given by using a generalized Lambert function $W_{\kappa,\gamma}$ as

$$S(z) = -1 - \frac{c}{zW_{\kappa,\gamma}(-\frac{c}{z})} - \frac{1-c\beta}{z} \qquad (z\in\mathbb{C}^+)$$

where

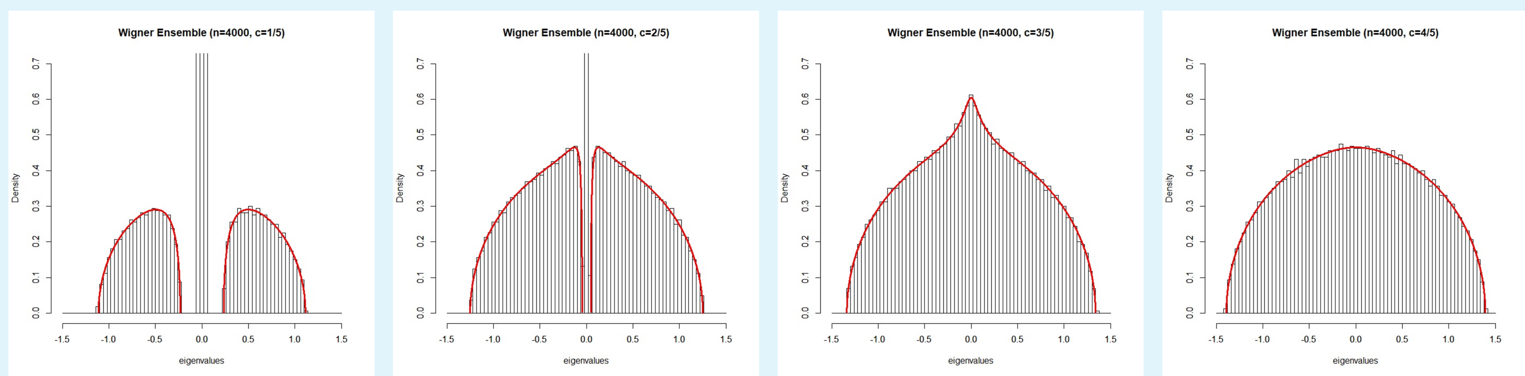
$$\kappa := \frac{1}{1-\alpha} \quad (\alpha\neq 1), \quad \kappa := +\infty \quad (\alpha = 1), \quad \gamma := 1-\beta.$$

Simulations

Simulations for Wigner Ensembles

The following pictures are histograms of eigenvalue distributions of a $U_n \in \mathcal{U}_n$ (Gaussian) for $n = 4000$ with $c = 0.2$, 0.4, 0.6 and 0.8. Red lines indicate the density functions in Theorem 1.

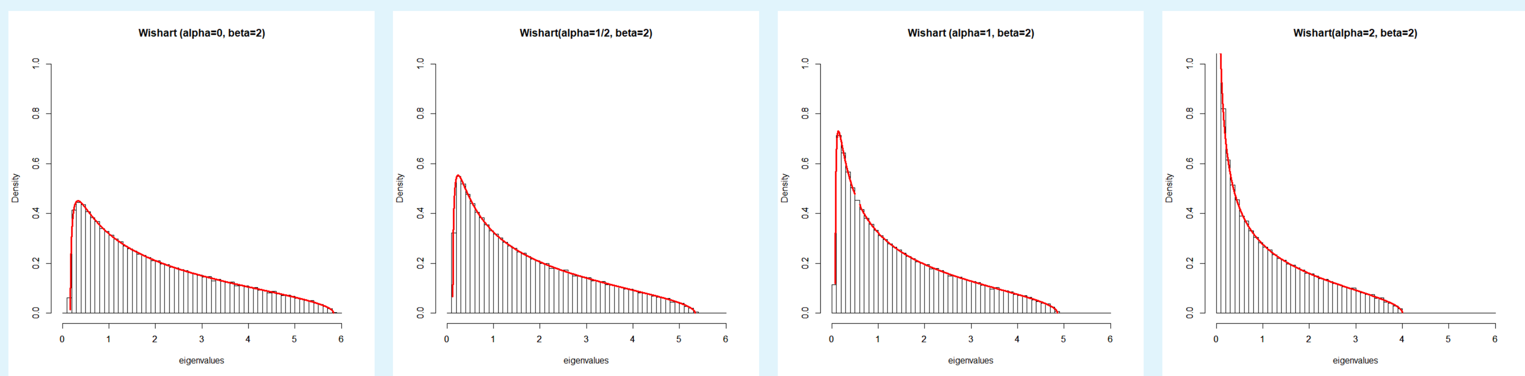
- $c = 0.2$ • $c = 0.4$ • $c = 0.6$ • $c = 0.8$



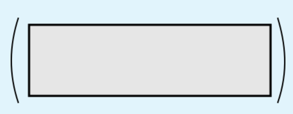
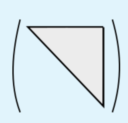
Simulations for Wishart Ensembles

The following pictures are histograms of eigenvalue distributions of an $X_n \in \mathcal{U}_n$ for $n = 4000$, $\beta = 2$ and $a_n = n-1$ with $\alpha = 0$ (the original Wishart), 0.5, 1 and 2 for Gaussian E_n. Red lines indicate the density functions obtained by Theorem 2.

- $\alpha = 0$ • $\alpha = 0.5$ • $\alpha = 1$ • $\alpha = 2$



Remarks

- Our results include the classical cases, that is, the Wigner's semicircle law for Wigner ensembles and the Marchenko-Pastur law for Wishart ensembles because we have $\mathcal{U}_n = \mathrm{Sym}(n,\mathbb{R})$ for $a_n = n-1$ and $b_n = 1$.
- Stieltjes transforms $S_\sigma(z)$ associated with variance profile σ is given as
$$S_\sigma(z) = \int_0^1 \eta_z(x)\,dx, \quad \text{where} \quad \eta_z(x) = \left(z + \int_0^1 \sigma(x,y)^2\eta_z(y)\,dy\right)^{-1}$$
- If $\gamma = 0$, then the generalized Lambert function $W_{\kappa,0}$ converges to the original Lambert function as $\kappa\to\infty$; in fact, we have $\lim_{\kappa\to\infty} f_{\kappa,0}(x) = xe^x$.
- Quadratic construction includes the following matrix spaces.
  - Spaces of rectangular matrices (Wishart), i.e. $E =$ 
  - Spaces of upper triangular matrices (Dykema–Haagerup [3]), i.e. $E =$ 

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