Mini-course on Fano Foliations

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Lecture 3: Classification of Fano foliations of large index

Mini-course on Fano Foliations

Joint with Stéphane Druel (CNRS/Université Claude Bernard Lyon 1)

- Lecture 0: Algebraicity of smooth formal schemes and applications to foliations
- Lecture 1: Definition, examples and first properties
- Lecture 2: Adjunction formula and applications
- Lecture 3: Classification of Fano foliations of large index

Classification of Fano manifolds

Theorem (Kollár-Miyaoka-Mori 1992)

For fixed n, Fano manifolds of dimension n form a bounded family

Classification in dimension ≤ 3 (Iskovskikh & Mori-Mukai 1977-1981)

Definition

The **index** of a Fano manifold X is

$$i(\mathcal{F}) \;:=\; max\left\{m \in \mathbb{Z} \;\middle|\; -K_X = mA,\; A \text{ ample}\right\}$$

Theorem (Kobayashi-Ochiai 1973)

- $i(X) \leq \dim(X) + 1$
- $i(X) = \dim(X) + 1 \iff X \cong \mathbb{P}^n$
- $i(X) = \dim(X) \iff X \cong Q^n \subset \mathbb{P}^{n+1}$

Classification of Fano manifolds

Theorem (Fujita 1982)

Classification when $i(X) = \dim(X) - 1$ (**del Pezzo manifolds**)

Theorem (Mukai 1992)

Classification when $i(X) = \dim(X) - 2$ (**Mukai manifolds**)

Theorem (Birkar 2016)

For singular Fano varieties, boundedness still holds if one suitably bounds the singularities (ϵ -lc)

Fano foliations

Problem

For fixed r and n, do Fano foliations of rank r on projective manifolds of dimension n form a bounded family?

Necessary condition (proved in lectures 0 and 1)

$\mathcal{F}$ Fano foliation $\implies$ $\exists$ subfoliation $\mathcal{G} \subset \mathcal{F}$ with algebraic and RC leaves

$\implies$ X is uniruled

Definition

The **index** of a Fano foliation $\mathcal{F}$ on complex projective manifold X is

$$i(\mathcal{F}) := max\left\{ m \in \mathbb{Z} \;\middle|\; -K_{\mathcal{F}} \sim_{\mathbb{Z}} mA, \; A \text{ ample } \right\}$$

Kobayashi-Ochiai theorem for foliations

Theorem (A.- Druel - Kovács 2008)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank r on a complex projective manifold X

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

Theorem (Wahl 1983)

X complex projective manifold

If T_X contains an ample line bundle, then $X \cong \mathbb{P}^n$

$\implies$ in the theorem we may assume that $r \geq 2$

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Proof.

Let $\mathcal{F} \subsetneq T_X$ be Fano foliation of rank $r \geq 2$ and index $i(\mathcal{F}) \geq r$

- Step 1. Show that $i(\mathcal{F}) = r$
- Step 2. Show that the leaves of $\mathcal{F}$ are algebraic
- Step 3. Show that the general log leaf $(F, \Delta) \cong (\mathbb{P}^r, H)$ (log canonical)
- Step 4. Using the common point, show that $X \cong \mathbb{P}^n$

Tool: rational curves on uniruled varieties

X complex projective manifold of dimension n

W dominating family of rational curves of minimal degree on X $(W \subset Chow(X))$

$x \in X$ general $\rightsquigarrow$ $W_x = \{[\ell] \in W \mid x \in \ell\}$ proper $\big(d = \dim(W_x)\big)$

Properties

- $\forall$ closed subset $Z \subset X$ with $\operatorname{codim}_X(Z) \geq 2$
  $\exists$ $\ell \in W$ such that $\ell \cap Z = \emptyset$
- For general $[\ell] \in W$, $T_{X|_\ell} \cong \underbrace{\mathcal{O}_{\mathbb{P}^1}(2)}_{= T_{\mathbb{P}^1}} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$

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Theorem (Cho-Miyaoka-Shepherd-Barron, Kebekus 2002)

$d = n-1 \iff X \cong \mathbb{P}^n \iff \exists x_0 \in X$ such that curves from W_{x_0} dominate X

Rationally connected quotients

X complex projective manifold

W dominating family of rational curves on X

Equivalence relation on X:

$$x \sim y \iff x \text{ and } y \text{ can be connected by a chain of cycles in } \overline{W}$$

$\exists$ dense open subset $X^\circ \subset X$ and proper morphism

$$\pi : X^\circ \to Y^\circ$$

whose fibers are equivalence classes

For general $[\ell] \in W$:

$$T_{X|_\ell} \cong \underbrace{\underbrace{\mathcal{O}_{\mathbb{P}^1}(2)}_{} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}}_{\subset (T_{X^\circ/Y^\circ})_{|\ell}} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$$

Rationally connected quotients

Remark

X complex projective manifold

W proper (unsplit) family of rational curves on X

(e.g., for some ample divisor A on X, $A \cdot \ell = 1$, $[\ell] \in W$)

$$x \sim y \iff x \text{ and } y \text{ can be connected by a chain of cycles in } W$$

$\exists$ dense open subset $X^\circ \subset X$ with $\mathrm{codim}_X(X \setminus X^\circ) \geq 2$ and equidimensional proper morphism onto normal variety

$$\pi : X^\circ \to Y^\circ$$

whose fibers are equivalence classes, reduced and irreducible

Kobayashi-Ochiai theorem for foliations

Theorem (A.- Druel - Kovács 2008)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank r on a complex projective manifold X

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

Proof.

Let $\mathcal{F} \subsetneq T_X$ be Fano foliation of rank $r \geq 2$ and index $i(\mathcal{F}) \geq r$

- Step 1. Show that $i(\mathcal{F}) = r$
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- Step 4. Using the common point, show that $X \cong \mathbb{P}^n$

Step 1. Show that $i(\mathcal{F}) = r$

Assumption: $-K_{\mathcal{F}} = i(\mathcal{F})A$, A ample and $i(\mathcal{F}) > r$

W dominating family of rational curves of minimal degree on X

with associated rationally connected quotient $\pi : X^\circ \to Y^\circ$

$[\ell] \in W$ general $\implies$ $\ell \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$ and

$\mathcal{F}_{|_\ell} \subset T_{X|_\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$

$\implies$ $\mathcal{F}_{|_\ell} \cong \underbrace{\mathcal{O}_{\mathbb{P}^1}(2)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ and $A \cdot \ell = 1$ (W unsplit)

$\implies$ $T_{X^\circ/Y^\circ} \subset \mathcal{F}_{|X^\circ}$

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \subset (T_{X^\circ/Y^\circ})_{|\ell} \subset \mathcal{F}_{|_\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1} \subset$$

$$\subset \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)} \cong T_{X|_\ell}$$

Step 1. Show that $i(\mathcal{F}) = r$

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$$\mathcal{F}_{|_\ell} \subset T_{X|_\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$$

$$\implies \mathcal{F}_{|_\ell} \cong \underbrace{\mathcal{O}_{\mathbb{P}^1}(2)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1} \text{ and } A \cdot \ell = 1 \ (W \text{ unsplit})$$

$$\implies T_{X^\circ/Y^\circ} \subset \mathcal{F}_{|X^\circ}$$

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \cong (T_{X^\circ/Y^\circ})_{|\ell} = \mathcal{F}_{|_\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1} \subset$$

$$\subset \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)} \cong T_{X|_\ell}$$

Step 1. Show that $i(\mathcal{F}) = r$

Assumption: $-K_{\mathcal{F}} = i(\mathcal{F})A$, A ample and $i(\mathcal{F}) > r$

W dominating family of rational curves of minimal degree on X

with associated rationally connected quotient $\pi : X^\circ \to Y^\circ$

Conclusion: $\mathcal{F}$ is induced by $\pi : X^\circ \to Y^\circ$

General log leaf $(F, \Delta) = (X_y, 0)$

Corollary (proved in lecture 2)

If $\mathcal{F}$ is an algebraically integrable Fano foliation on a complex projective manifold, then $\Delta \neq 0$.

Contradiction! □

Kobayashi-Ochiai theorem for foliations

Theorem (A.- Druel - Kovács 2008)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank r on a complex projective manifold X

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Proof.

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- Step 2. Show that the leaves of $\mathcal{F}$ are algebraic
- Step 3. Show that the general log leaf $(F, \Delta) \cong (\mathbb{P}^r, H)$ (log canonical)
- Step 4. Using the common point, show that $X \cong \mathbb{P}^n$

Step 2. Show that leaves are algebraic

Assumption: $-K_{\mathcal{F}} = rA$, A ample

W dominating family of rational curves of minimal degree on X

$W \rightsquigarrow \alpha \in N_1(X)$ movable curve class $\rightsquigarrow \mu_\alpha(\bullet) = \frac{\det(\bullet)\cdot\alpha}{\text{rank}(\bullet)}$

The **Harder-Narasimhan filtration** of $\mathcal{F}$:

$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{F}$

$(\ \mu_\alpha(\mathcal{F}_1) > \mu_\alpha(\mathcal{F}_2) > \cdots > \mu_\alpha(\mathcal{F}_k) \geq 1 \ )$

Theorem (proved in lectures 0 and 1)

$\mathcal{F}_1$ has algebraic (and RC) leaves

Case 1. $\mathcal{F} = \mathcal{F}_1$ is μ_α-semistable $\implies$ $\mathcal{F}$ has algebraic leaves

Case 2. $\mathcal{F}_1 \neq \mathcal{F}$ $\implies$ $\mu_\alpha(\mathcal{F}_1) > 1$

Step 2. Show that leaves are algebraic

Case 2. $\mathcal{F}_1 \subsetneq \mathcal{F}$ with $\mu_\alpha(\mathcal{F}_1) = \frac{\det(\mathcal{F}_1)\cdot\alpha}{\mathrm{rank}(\mathcal{F}_1)} > 1 \implies$ (as in step 1)

- W unsplit
- $\mathcal{F}_1$ has rank $r-1$
- $\mathcal{F}_1$ is induced by the rationally connected quotient associated to W

$$\pi : X^\circ \to Y^\circ$$

($\mathrm{codim}_X(X \setminus X^\circ) \geq 2$ and π equidimensional and proper with reduced and irreducible fibers onto normal variety)

$\implies$ $\mathcal{F} = \pi^*\mathcal{G}$ for $\mathcal{G} \subset T_{Y^\circ}$ foliation of rank 1

$$\boxed{K_{\mathcal{F}} = K_{X^\circ/Y^\circ} + \pi^* K_{\mathcal{G}}}$$

Step 2. Show that leaves are algebraic

$X^\circ \subset X$ open subset with $\mathrm{codim}_X(X \setminus X^\circ) \geq 2$

$\pi : X^\circ \to Y^\circ$ equidimensional and proper with reduced fibers

$\mathcal{G} \subset T_{Y^\circ}$ foliation of rank 1

$\mathcal{F} = \pi^*\mathcal{G} \quad \rightsquigarrow \quad \boxed{-K_{\mathcal{F}} = -K_{X^\circ/Y^\circ} - \pi^* K_{\mathcal{G}}}$

$\tilde{C} \subset X$ general complete intersection curve $\implies \tilde{C} \subset X^\circ$

$C = \pi(\tilde{C}) \subset Y^\circ$ (we may assume it is smooth) and $X_C = \pi^{-1}(C)$

$\pi_C : X_C \to C$ equidimensional and proper with reduced fibers

$$\underbrace{(-K_{\mathcal{F}})_{|X_C}}_{\text{ample}} = \underbrace{-K_{X_C/C}}_{\text{cannot be ample}} - \pi^*(K_{\mathcal{G}|C})$$

$\implies \quad -K_{\mathcal{G}} \cdot C > 0$

$\implies \quad$ leaves of $\mathcal{G}$ are algebraic $\square$

Kobayashi-Ochiai theorem for foliations

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Step 3. Show that $(F,\Delta) \cong (\mathbb{P}^r, H)$

Assumption: $-K_{\mathcal{F}} = rA$, A ample $+$ leaves are algebraic

$$\rightsquigarrow \quad (F,\Delta) \text{ general log leaf } \big(\Delta \neq 0\big)$$

Adjunction theory: To describe a polarized variety (Y, L) by studying

$$K_Y + mL\ , \quad m \geq 1 \quad (\text{ adjunction divisors })$$

Example (Fujita 1988)

$K_Y + \dim(Y)L$ not pseudo-effective $\implies (Y, L) \cong (\mathbb{P}^n, H)$

In our case: $(Y, L) = (F, A_F)$

$$\boxed{K_F + \Delta \ \sim\ (K_{\mathcal{F}})_{|F} \ \sim\ -rA_F}$$

$\implies \quad K_F + rA_F \ \sim\ -\Delta$ is not pseudo-effective

$\implies \quad (F, A_F) \ \cong\ (\mathbb{P}^r, H)$ and $\Delta \sim H$ □

Kobayashi-Ochiai theorem for foliations

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Step 4. Show that $X \cong \mathbb{P}^n$

Assumption: $-K_{\mathcal{F}} = rA$, A ample $+$ leaves are algebraic
General log leaf $(F, \Delta) \cong (\mathbb{P}^r, H)$ and $A_F \sim H$

$\ell \subset F \cong \mathbb{P}^r \rightsquigarrow W$ dominating (unsplit) family of rational curves on X

Corollary (proved in Lecture 2)

$\mathcal{F}$ algebraically integrable Fano foliation on a complex projective manifold

If the general log leaf (F, Δ) is log canonical, then there is a common point in the closure of a general leaf.

$x_0 \in X$ commont point in the closure of a general leaf $F \cong \mathbb{P}^r$

Curves from W_{x_0} dominate $X \implies X \cong \mathbb{P}^n$ □

del Pezzo foliations

Theorem (A.- Druel - Kovács 2008)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank r on a complex projective manifold X

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

Definition

A Fano foliation $\mathcal{F} \subsetneq T_X$ of rank r on a complex projective manifold X is a **del Pezzo foliation** if $i(\mathcal{F}) = r - 1$.

del Pezzo foliations

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A Fano foliation $\mathcal{F} \subsetneq T_X$ of rank r on a complex projective manifold X is a **del Pezzo foliation** if $i(\mathcal{F}) = r - 1$.

Theorem (A.- Druel 2013)

If $\mathcal{F}$ is a del Pezzo foliation on a complex projective manifold X, then

- either $X \cong \mathbb{P}^n$ and $\exists\ \varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r+1}$ such that $\mathcal{F} = \varphi^*\mathcal{C}$ for $\mathcal{C} \cong \mathcal{O} \subset T_{\mathbb{P}^{n-r+1}}$, or
- $\mathcal{F}$ is algebraically integrable

Problem

Classification of del Pezzo foliations

Classification of del Pezzo foliations

Theorem (A.- Druel 2016, A. 2018)

Classification of log leaves (F, Δ) of del Pezzo foliations on complex projective manifolds:

1. $(F, \Delta) \cong (\mathbb{P}^r, Q^{r-1})$
2. $(F, \Delta) \cong (Q^r, H)$
3. $(F, \Delta) \cong (\mathbb{P}^2, \ell)$
4. $F \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ + classification of $\mathcal{E}$ and description of Δ ($r \leq 3$)
5. (F, Δ) is a cone over $(C_d, p_1 + p_2)$, where C_d is rational normal curve of degree d in $\mathbb{P}^d$
6. (F, Δ) is a cone over (4)

Classification of del Pezzo foliations

Theorem (A.- Druel 2013, 2016, Figueredo 2019)

$\mathcal{F}$ del Pezzo foliation of rank $r \geq 3$ on complex projective manifold $X \not\cong \mathbb{P}^n$
Suppose that the general log leaf (F, Δ) is log canonical. Then

- either $X \cong Q^n$ and $\mathcal{F}$ is induced by a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n-r}$
- or $r = 3$ and $X \cong \mathbb{P}_{\mathbb{P}^k}(\mathcal{E})$ (+ classification of $\mathcal{E}$ and $\mathcal{F}$)

Problem

Classification of del Pezzo foliations of rank $r = 2$

Thank you!