Mini-course on Fano Foliations

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Lecture 1: Definition, examples and first properties

Mini-course on Fano Foliations

Joint with Stéphane Druel (CNRS/Université Claude Bernard Lyon 1)

- Lecture 0: Algebraicity of smooth formal schemes and applications to foliations
- Lecture 1: Definition, examples and first properties
- Lecture 2: Adjunction formula and applications
- Lecture 3: Classification of Fano foliations of large index

Motivation from the MMP

$K_X > 0$	$K_X = 0$	$K_X < 0$
		Fano varieties

Special geometric properties

Fano manifolds are rationally connected (RC)

Classification

Classification of Fano manifolds with large index

Fano foliations

X normal complex projective variety

Foliation $\mathcal{F} \subsetneq T_X$ on X

- **saturated** ($T_X/\mathcal{F}$ torsion-free)
- **integrable** ($[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$)

$\mathrm{Sing}(\mathcal{F}) \subset X$ degeneracy locus of the map $\mathcal{F} \hookrightarrow T_X$

The **canonical class** of $\mathcal{F}$: $K_{\mathcal{F}} \in C\ell(X)$

$$\mathcal{O}_X(K_{\mathcal{F}}) \cong \big(\det(\mathcal{F})\big)^{\vee}$$

$\mathcal{F} \hookrightarrow T_X \quad \rightsquigarrow \quad \Omega_X^r \to \mathcal{O}_X(K_{\mathcal{F}})$

$\mathcal{I}_{\mathrm{Sing}(\mathcal{F})}$ is the image of the induced map $\big(\Omega_X^r(-K_{\mathcal{F}})\big)^{\vee\vee} \to \mathcal{O}_X$

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Definition

$\mathcal{F}$ is **Fano** if $-K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier and ample

Early examples: foliations on $\mathbb{P}^n$

$\mathcal{F} \subsetneq T_{\mathbb{P}^n}$ foliation of rank r on $\mathbb{P}^n$

$d = \deg(\mathcal{F})$ **degree** of $\mathcal{F}$

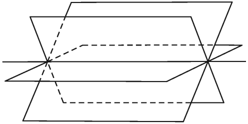
$L = \mathbb{P}^{n-r} \subset \mathbb{P}^n$ general linear subspace

$D_{\mathcal{F}} \subset L = \mathbb{P}^{n-r}$ tangency hypersurface

$d = \deg(D_{\mathcal{F}}) \geq 0$

Early problem: Classification of (codim 1) foliations of low degree on $\mathbb{P}^n$

Foliations of degree 0 on $\mathbb{P}^n$



$\mathcal{F}$: r-planes on $\mathbb{P}^n$ containing a fixed $L_0 = \mathbb{P}^{r-1}$

$\mathbb{P}^{n-r} \subset \mathbb{P}^n$ general linear subspace - everywhere transvere to $\mathcal{F}$

$\deg(\mathcal{F}) = 0$

$\mathcal{F}$ is induced by the linear projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$ from L_0

$\mathcal{F} \ = \ \ker(d\pi) \ \cong \ \mathcal{O}(1)^{\oplus r}$ and $-K_{\mathcal{F}} = rH$

Theorem (Jouanolou 1979, Déserti-Cerveau 2005)

These are the only foliations of degree 0 on $\mathbb{P}^n$

The index of a Fano foliations

Definition

The **index** of a Fano foliation $\mathcal{F}$ on complex projective manifold X is

$$i(\mathcal{F}) \ := \ max\{m \in \mathbb{Z} \mid -K_{\mathcal{F}} \sim_{\mathbb{Z}} mA, \ A \text{ ample } \}$$

Example

$\mathcal{F} \ \cong \ \mathcal{O}(1)^{\oplus r}$ foliation of degree 0 on $\mathbb{P}^n \ \Longrightarrow \ i(\mathcal{F}) \ = \ r$

$$\mathcal{F} \hookrightarrow T_X \ \rightsquigarrow \ \mathcal{O}_X(-K_{\mathcal{F}}) \to (\Omega_X^r)^{\vee} \cong \Omega_X^{n-r}(-K_X) \ \rightsquigarrow$$

$$\rightsquigarrow \ \boxed{0 \neq \omega \in H^0\big(X, \Omega_X^{n-r}(-K_X + K_{\mathcal{F}})\big)}$$

Example (Fano foliations on $\mathbb{P}^n$)

$\mathcal{F} \subsetneq T_{\mathbb{P}^n}$ Fano foliation on $\mathbb{P}^n$ of rank r and index i

$$\boxed{\omega \in H^0\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-r}(n+1-i)\big)}$$

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Example

$\mathcal{F} \ \cong \ \mathcal{O}(1)^{\oplus r}$ foliation of degree 0 on $\mathbb{P}^n \quad \Longrightarrow \quad i(\mathcal{F}) \ = \ r$

Example (Fano foliations on $\mathbb{P}^n$)

$\mathcal{F} \subsetneq T_{\mathbb{P}^n}$ Fano foliation of rank r and index i

$$\boxed{\omega \in H^0\big(\mathbb{P}^n, \Omega^{n-r}_{\mathbb{P}^n}(n+1-i)\big)}$$

$L = \mathbb{P}^{n-r} \subset \mathbb{P}^n$ general linear subspace
$\omega_{|L} \ \in \ H^0\big(\mathbb{P}^{n-r}, \Omega^{n-r}_{\mathbb{P}^{n-r}}(n+1-i)\big) \ = \ H^0\big(\mathbb{P}^{n-r}, \mathcal{O}_{\mathbb{P}^{n-r}}(r-i)\big)$

$$\boxed{i \ = \ r-d}$$

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Example (Fano foliations on $\mathbb{P}^n$)

$\mathcal{F} \subsetneq T_{\mathbb{P}^n}$ Fano foliation of rank r, index i, and degree d

$$\boxed{i \ = \ r - d \ \leq r}$$

Theorem (A.- Druel - Kovács 2008)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank r on a complex projective manifold X

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

The index of a Fano foliations

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$$\boxed{i \ = \ r - d \ \leq r}$$

Theorem (A.- Druel 2014, Höring 2014)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank r on a normal complex projective variety X

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X$ is a generalized cone

Foliations of degree 1 on $\mathbb{P}^n$ (index $r-1$)

Construction (Algebraically integrable foliation)

$\varphi : X \dashrightarrow Y$ dominant rational map with connected fibers

$\varphi^\circ : X^\circ \to Y^\circ$ equidimensional morphism

$X^\circ \subset X$ open subset with $\mathrm{codim}_X(X \setminus X^\circ) \geq 2$

$$\rightsquigarrow \mathcal{F} \subset T_X \quad \text{saturation of } \ker(d\varphi^\circ) \text{ in } T_X$$

$$\boxed{K_{\mathcal{F}} = K_{X/Y} - R(\varphi)}$$

$R(\varphi)$ ramification divisor of φ

Example 1:

$$\varphi : \begin{array}{ccc} \mathbb{P}^n & \dashrightarrow & \mathbb{P}(1^{n-r}, 2) \\ (x_0 : \cdots : x_n) & \longmapsto & (L_1 : \cdots : L_{n-r} : Q) \end{array}$$

$$\boxed{-K_{\mathcal{F}} = (r-1)H}$$

Foliations of degree 1 on $\mathbb{P}^n$ (index $r-1$)

Construction (Pullback foliations)

$\varphi: X \dashrightarrow Y$ dominant rational map with connected fibers

$\varphi^\circ: X^\circ \to Y^\circ$ equidimensional morphism

$X^\circ \subset X$ open subset with $\mathrm{codim}_X(X \setminus X^\circ) \geq 2$

$\mathcal{G} \subset T_Y$ foliation on Y

$\rightsquigarrow$ **Pullback foliation** $\mathcal{F} = \varphi^*\mathcal{G}$

$\mathcal{F}$ is the saturation of $(d\varphi^\circ)^{-1}(\mathcal{G}_{|Y^\circ})$ in T_X

$$\boxed{K_\mathcal{F} = K_{X/Y} + \varphi^* K_\mathcal{G} - R(\varphi)^\mathcal{G}}$$

Foliations of degree 1 on $\mathbb{P}^n$ (index $r-1$)

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$\mathcal{G} \subset T_Y$ foliation on Y $\rightsquigarrow$ $\mathcal{F} = \varphi^*\mathcal{G}$

$$\boxed{K_{\mathcal{F}} = K_{X/Y} + \varphi^* K_{\mathcal{G}} - R(\varphi)^{\mathcal{G}}}$$

Example 2: $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r+1}$ linear projection

$\mathcal{C} \subset T_{\mathbb{P}^{n-r+1}}$ foliation induced by a global vector field ($K_{\mathcal{C}} = 0$)

$\rightsquigarrow$ $\mathcal{F} = \pi^*\mathcal{C} \subset T_{\mathbb{P}^n}$

$$\boxed{-K_{\mathcal{F}} = (r-1)H}$$

If $\mathcal{C}$ is general, then $\mathcal{C}$ and $\mathcal{F}$ have transcendental leaves

Foliations of degree 1 on $\mathbb{P}^n$ (index $r-1$)

Theorem (Jouanolou 1979, Loray-Pereira-Touzet 2018)

There are 2 types of foliations of degree 1 on $\mathbb{P}^n$ (index $r-1$) :

- $\mathcal{F}$ is induced by $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^{n-r}, 2)$
- $\exists\ \varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r+1}$ and such that $\mathcal{F} = \varphi^*\mathcal{C}$ for $\mathcal{C} \subset T_{\mathbb{P}^{n-r+1}}$ foliation of rank 1 induced by a global vector field

Theorem (Cerveau - Lins Neto 1996)

There are 6 types of codimension 1 foliations of degree 2 on $\mathbb{P}^n$

(rank $r = n-1$ and index $i = r-2$)

More examples: foliations on hypersurfaces

Construction (Restrictions of foliations)

$\mathcal{F}$ foliation of codimension $q \geq 1$ on smooth projective variety X

$Y \subset X$ smooth subvariety generically transverse to $\mathcal{F}$

$\mathcal{F}$ restricts to a foliation $\mathcal{F}_Y$ of codimension q on Y

$\mathcal{F} \subset T_X \;\leftrightsquigarrow\; \omega \in H^0\big(X, \Omega_X^q(-K_X + K_{\mathcal{F}})\big)$

$\mathcal{F}_Y \subset T_Y \;\leftrightsquigarrow\; \omega_Y \in H^0\big(Y, \Omega_Y^q(-K_Y + K_{\mathcal{F}_Y})\big)$

$$(-K_X + K_{\mathcal{F}})_{|Y} \;=\; -K_Y + K_{\mathcal{F}_Y} + B \qquad (\ B \geq 0\ )$$

$$\boxed{K_{\mathcal{F}_Y} \;=\; c_1\big(N_{Y/X}\big) + (K_{\mathcal{F}})_{|Y} - B}$$

Fano foliations on hypersurfaces

Construction (Restrictions of foliations)

$\mathcal{F}$ foliation of codimension $q \geq 1$ on smooth projective variety X

$Y \subset X$ smooth subvariety generically transverse to $\mathcal{F}$

$\mathcal{F}$ restrics to a foliation $\mathcal{F}_Y$ of codimension q on Y

$$\boxed{K_{\mathcal{F}_Y} = c_1\big(N_{Y/X}\big) + (K_{\mathcal{F}})_{|Y} - B} \qquad (\, B \geq 0 \,)$$

Example ($X = \mathbb{P}^n$, $Y \subset \mathbb{P}^n$ hypersurface of degree $d \geq 2$)

$\mathcal{F} \subset T_{\mathbb{P}^n}$ Fano foliation of index $i \leq r$

$-K_{\mathcal{F}_Y} \geq (i - d)\, H_{|Y}$

More examples: foliations on hypersurfaces

Construction (Restrictions of foliations)

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Example ($X = \mathbb{P}^n$, $Y \subset \mathbb{P}^n$ hypersurface of degree $d \geq 2$)

$\mathcal{F} \subset T_{\mathbb{P}^n}$ Fano foliation of index $i = r$

$-K_{\mathcal{F}_Y} = (i - d)\, H_{|Y}$

$(\ h^0\big(Y, \Omega_Y^q(q+1-d)\big) = 0\ )$

Algebraicity properties of Fano foliations

Theorem (Campana - Păun 2019)

X normal projective $\mathbb{Q}$-factorial variety, $\alpha \in N_1(X)_{\mathbb{R}}$ movable curve class, $\mathcal{G} \subset T_X$ foliation on X with $\mu_\alpha^{\min}(\mathcal{G}) > 0$.

Then $\mathcal{G}$ has algebraic and RC leaves.

$\mu_\alpha(\bullet) = \frac{\det(\bullet)\cdot\alpha}{\operatorname{rank}(\bullet)}$

$\mu_\alpha^{\min}(\mathcal{G}) = \inf\left\{\mu_\alpha(\mathcal{Q}) \,|\, \mathcal{Q} \neq 0 \text{ is a torsion-free quotient of } \mathcal{G}\right\}$

Corollary

X normal projective $\mathbb{Q}$-factorial variety, $\mathcal{F} \subset T_X$ Fano foliation.

Then $\exists$ subfoliation $\mathcal{G} \subset \mathcal{F}$ with algebraic and RC leaves.

Proof of corollary

X normal projective $\mathbb{Q}$-factorial variety, $\alpha \in N_1(X)_{\mathbb{R}}$ movable curve class, $\mathcal{F}$ torsion free sheaf on X

The Harder-Narasimhan filtration:

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{F}$$

with $\mathcal{Q}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ μ_α-semistable, and

$$\mu_\alpha(\mathcal{Q}_1) > \mu_\alpha(\mathcal{Q}_2) > \cdots > \mu_\alpha(\mathcal{Q}_k)$$

$\mu_\alpha^{\min}(\mathcal{F}_i) \; = \; \mu_\alpha(\mathcal{Q}_i)$

$\mathcal{F}$ foliation $\quad \Longrightarrow \quad \mathcal{F}_i$ foliation whenever $\mu_\alpha(\mathcal{Q}_i) \geq 0$

$\mathcal{F}$ Fano foliation $\quad \Longrightarrow \quad \mu_\alpha(\mathcal{F}) > 0 \quad \Longrightarrow \quad \mu_\alpha(\mathcal{Q}_1) > 0$

Proof of corollary

The Harder-Narasimhan filtration:

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with $\mathcal{Q}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ μ_α-semistable, and

$$\mu_\alpha(\mathcal{Q}_1) > \mu_\alpha(\mathcal{Q}_2) > \cdots > \mu_\alpha(\mathcal{Q}_k)$$

$\mu_\alpha^{\min}(\mathcal{F}_i) \ = \ \mu_\alpha(\mathcal{Q}_i)$

$\mathcal{F}$ foliation $\implies$ $\mathcal{F}_i$ foliation whenever $\mu_\alpha(\mathcal{Q}_i) \geq 0$

$\mathcal{F}$ Fano foliation $\implies$ $\mu_\alpha(\mathcal{F}) > 0$ $\implies$ $\mu_\alpha(\mathcal{Q}_1) > 0$

Theorem (Campana - Păun 2019)

X normal projective $\mathbb{Q}$-factorial variety, $\alpha \in N_1(X)_\mathbb{R}$ movable curve class, $\mathcal{F}_1 \subset T_X$ foliation on X with $\mu_\alpha^{\min}(\mathcal{F}_1) > 0$.

Then $\mathcal{F}_1$ has algebraic and RC leaves. □

Proof of corollary

The Harder-Narasimhan filtration:

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = \mathcal{F}$$

with $\mathcal{Q}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ μ_α-semistable, and

$$\mu_\alpha(\mathcal{Q}_1) > \mu_\alpha(\mathcal{Q}_2) > \cdots > \mu_\alpha(\mathcal{Q}_k)$$

$\mu_\alpha^{\min}(\mathcal{F}_i) \; = \; \mu_\alpha(\mathcal{Q}_i)$

$\mathcal{F}$ foliation $\quad \Longrightarrow \quad \mathcal{F}_i$ foliation whenever $\mu_\alpha(\mathcal{Q}_i) \geq 0$

$\mathcal{F}$ Fano foliation $\quad \Longrightarrow \quad \mu_\alpha(\mathcal{F}) > 0 \quad \Longrightarrow \quad \mu_\alpha(\mathcal{Q}_1) > 0$ $\square$

Remark

$s := \max \{ \, 1 \leq i \leq k \,|\, \mu_\alpha(\mathcal{Q}_i) > 0 \, \} \geq 1$

Then $\mathcal{F}_1, \ldots, \mathcal{F}_s$ have algebraic and RC leaves.

The algebraic rank of a foliation

X normal complex projective variety

$\mathcal{F} \subset T_X$ foliation on X

$\exists \quad \varphi : X \dashrightarrow Y$ dominant rational map with connected fibers

$\exists \ \mathcal{G}$ purely transcendental foliation on Y

$\mathcal{F} = \varphi^* \mathcal{G}$

Definition (algebraic rank)

$rk^{alg}(\mathcal{F}) := \dim(X) - \dim(Y)$

Bounding the algebraic rank

Theorem (A.-Druel 2019)

X complex projective manifold, $\mathcal{F} \subsetneq T_X$ Fano foliation of index $i(\mathcal{F})$

- $rk^{alg}(\mathcal{F}) \;\geq\; i(\mathcal{F})$
- $rk^{alg}(\mathcal{F}) \;=\; i(\mathcal{F}) \;\Longrightarrow\; X \cong \mathbb{P}^n$

  $\exists\, \varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ and $\mathcal{G}$ purely transcendental foliation on $\mathbb{P}^m$ such that

$$K_{\mathcal{G}} \equiv 0 \quad \text{and} \quad \mathcal{F} = \varphi^*\mathcal{G}$$

del Pezzo foliations

Theorem (A.- Druel - Kovács 2008)

$\mathcal{F} \subsetneq T_X$ Fano foliation of rank r on a complex projective manifold X

- $i(\mathcal{F}) \leq r$
- $i(\mathcal{F}) = r \implies X \cong \mathbb{P}^n$

Definition

A Fano foliation $\mathcal{F} \subsetneq T_X$ of rank r on a complex projective manifold X is a **del Pezzo foliation** if $i(\mathcal{F}) = r - 1$.

The algebraic rank of del Pezzo foliations

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A Fano foliation $\mathcal{F} \subsetneq T_X$ of rank r on a complex projective manifold X is a **del Pezzo foliation** if $i(\mathcal{F}) = r - 1$.

Theorem (A.-Druel 2019)

X complex projective manifold, $\mathcal{F} \subsetneq T_X$ Fano foliation of index $i(\mathcal{F})$

- $rk^{alg}(\mathcal{F}) \ \geq \ i(\mathcal{F})$
- $rk^{alg}(\mathcal{F}) \ = \ i(\mathcal{F}) \ \Longrightarrow \ X \cong \mathbb{P}^n$

Corollary (A.- Druel 2013)

A del Pezzo foliation $\mathcal{F}$ on a complex projective manifold $X \not\cong \mathbb{P}^n$ is algebraically integrable.

del Pezzo foliations

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A Fano foliation $\mathcal{F} \subsetneq T_X$ of rank r on a complex projective manifold X is a **del Pezzo foliation** if $i(\mathcal{F}) = r - 1$.

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Problem

Classification of del Pezzo foliations

Thank you!