

Complete holomorphic vector fields and their singular points

Adolfo Guillot
UNAM, Mexico
adolfo.guillot@im.unam.mx

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Part IV

The birational point of view

The birational point of view: reduced vector fields

Reduced vector fields:

- The foliation \mathcal{F} is reduced (Seidenberg)
- The union of the curve of zeros and
 - ▶ the leaf of \mathcal{F} through p , if $p \notin \text{sing}(\mathcal{F})$
 - ▶ the separatrices of \mathcal{F} through p , if $p \in \text{sing}(\mathcal{F})$

has normal crossings.

The birational point of view: normal forms

Theorem (G.-Rebello)

Let M be a surface, X be a **reduced** holomorphic semicomplete vector field on M . Let p be a point where $X(p) = 0$. Either

- p is an isolated non-degenerate singularity of X or,

up to multiplication by a non-vanishing function, X is either:

- $x(1 + \lambda y)\partial/\partial x + y^2\partial/\partial y$, $\lambda \in \mathbb{Z}$,
- $x^p y^q (mx\partial/\partial x - ny\partial/\partial y)$, $pm - qn = 1$ ($m, n \in \mathbb{Z}$, $m, n \geq 0$)
- $(x^n y^m)^r [(mx + \dots)\partial/\partial x - (ny + \dots)\partial/\partial y]$, $(m, n) = 1$, $r \geq 1$
- $x^r y^q \partial/\partial x$ with $r \in \{0, 1, 2\}$, $q \geq 0$.

One result

Theorem (G., 2014)

Let V be a two-dimensional normal irreducible Stein space, X a complete holomorphic vector field on V , p singular point of V , isolated zero of X . Either:

- *p is a weighted homogeneous singularity (X generates the weighted homotheties); or*
- *p is a cyclic quotient (Hirzebruch-Jung) singularity (X is the quotient of a non-degenerate vector field).*

... a flow in the quasihomogeneous singularity E_8

Let

$$V = \{x^2 + y^3 + z^5 = 0\}.$$

It is preserved by

$$15x \frac{\partial}{\partial x} + 10y \frac{\partial}{\partial y} + 6z \frac{\partial}{\partial z},$$

whose flow in time t is, for $\alpha = e^t$,

$$(x, y, z) \mapsto (\alpha^{15}x, \alpha^{10}y, \alpha^6z)$$

...flows in a cyclic quotient singularity

Complete vector fields in \mathbb{C}^2 :

$$z \frac{\partial}{\partial z} + (mw + z^m) \frac{\partial}{\partial w},$$

$$\lambda z \frac{\partial}{\partial z} + \mu w \frac{\partial}{\partial w}.$$

Symmetry: for $\omega^n = 1$, $(n, m) = 1$,

$$(z, w) \mapsto (\omega z, \omega^m w).$$

The quotient is the *Hirzebruch-Jung* or *cyclic quotient* surface singularity $A_{n,m}$. It has complete vector fields.

Another result

Dloussky, Oeljeklaus and Toma completed the classification of compact complex surfaces admitting holomorphic vector fields (2000).

Proposition (G., 2014)

Let S be a minimal compact complex surface, X a holomorphic vector field on S . Either:

- *X has no zeros,*
- *the flow of X factors through an action of \mathbb{C}^* ,*
- *X has a first integral,*
- *X is a Kato surface, or*
- *there is a divisor of elliptic fibre type invariant by X .*

Some questions

- Say something about complete polynomial vector fields in \mathbb{C}^3 .
- Say something about complete holomorphic vector fields in \mathbb{C}^2 .
- Let X be a germ of semicomplete vector field in $(\mathbb{C}^3, 0)$ with an isolated singularity. Does it have a separatrix? Can its second jet be trivial?

References



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Vector fields, separatrices and Kato surfaces.

Ann. Inst. Fourier (Grenoble), 64(3):1331–1361, 2014.



Adolfo Guillot and Julio Rebelo.

Semicomplete meromorphic vector fields on complex surfaces.

J. Reine Angew. Math., 667:27–65, 2012.

The second one contains the discussion about the birational normal forms for (holomorphic and meromorphic!) semicomplete vector fields on surfaces.

The results about flows on singular Stein surfaces appear in the first one.