Complete holomorphic vector fields and their singular points

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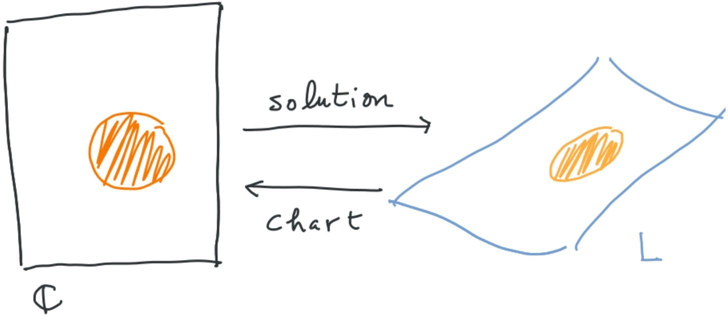
Recap

A holomorphic vector field X on a complex manifold induces a holomorphic foliation (away from the zeros of X). X gives each leaf a time form dT —$dT(X) \equiv 1$— and a translation structure.



Complete and semicomplete vector fields

Complete vector field

Each one-dimensional orbit is of the form $\mathbb{C}/G$ ($G \subset \mathbb{C}$, discrete).

Semicomplete vector field

- For each leaf L there exists $\Omega_L \subset \mathbb{C}$ and a Galois covering map $\phi : \Omega_L \to L$ whose deck transformations are translations.
- For each leaf L, each open path $\gamma : [0, 1] \to L$, $\int_\gamma dT_L \neq 0$.

Some facts

- Complete vector fields are semicomplete
- The restriction of a semicomplete vector field is still semicomplete
- It makes sense to speak about germs of semicomplete vector fields
- Germs of semicomplete vector fields give the local models for complete vector fields.

Part III

Non isolated singularities

Vector fields vs. foliations

$$X = f(x,y)\underbrace{\left(A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}\right)}_{\text{foliation}},\ (A,B) = 1$$

f defines a curve of singularities (non-isolated singularities).

Affine structures on curves

Affine group

$$\mathrm{Aff}(\mathbb{C}) = \{z \mapsto az + b\}$$

Affine structure on a curve L:

- atlas for L with changes of coordinates in $\mathrm{Aff}(\mathbb{C})$;
- cover $\{U_i\}$, $L = \cup U_i$, X_i vector field on U_i without zeros, $X_i = c_{ij} X_j$, c_{ij} constant; or
- a vector field X in $\widetilde{L}$ such that deck transformations act by multiplying X by constants.

Example

Translation structures!

Renormalization and the limit affine structure

The asymptotic affine structure

The affine classes of the translation structures on the leafs of

$$X = h(x,y)x^n \frac{\partial}{\partial y}$$

with $h(0) \neq 0$, extend to $x = 0$ in a unique way.

First integrals of X: $f(x)$.
If $X' = f(x)x^n \partial/\partial y$ does not vanish at $x = 0$ then $f = g(x)x^{-n}$ with $g(0) \neq 0$:

$$X'|_{x=0} = g(0)\frac{\partial}{\partial y}.$$

The restrictions of all these vector fields differ by a constant: there is a well-defined affine structure!

Uniformizable affine structures

L curve w/affine structure

The affine structure is uniformizable if it satisfies one of the following two equivalent properties:

- L is the quotient of $\Omega_L \subset \mathbb{C}$ under a subgroup of $\mathrm{Aff}(\mathbb{C})$.
- for every open path $\gamma : [0,1] \to L$, the development of the affine structure along γ maps the endpoints to different points.

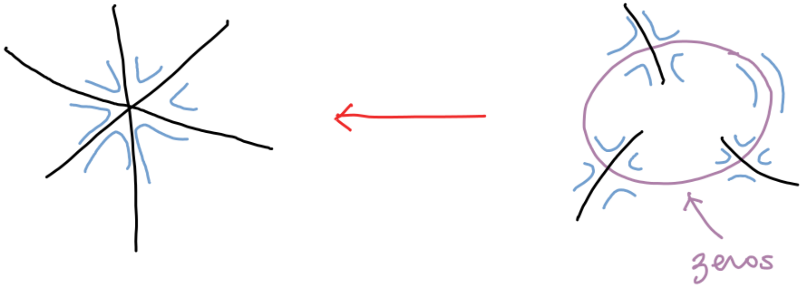
If the affine structure is the translation structure induced by a vector field, this is the same as semicompleteness.

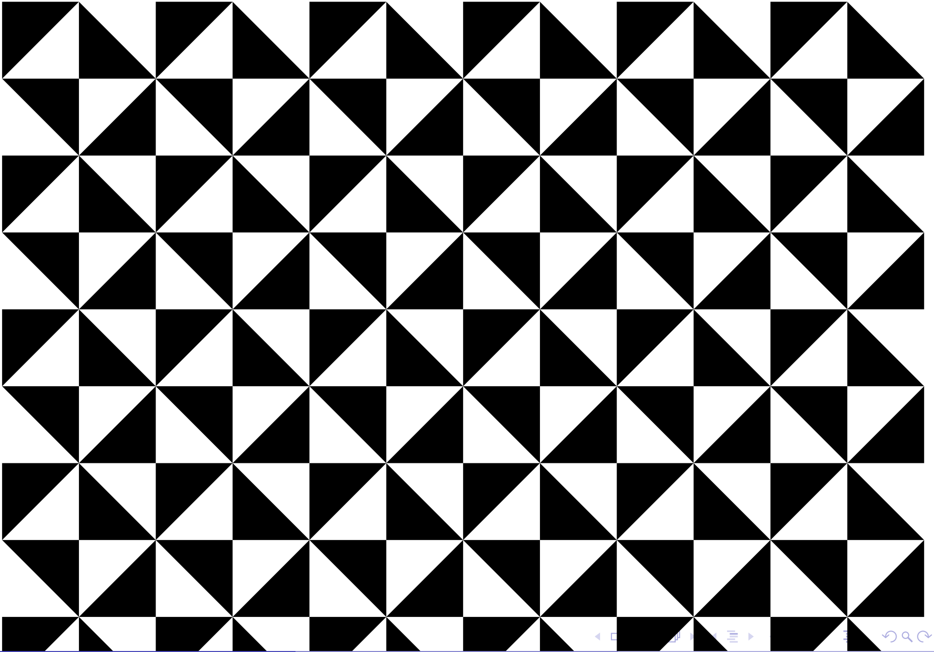
Back to semicompleteness!

If X is semicomplete vector field on M, C a component of the curve zeros of X **invariant** by the foliation, then the affine structure on C is uniformizable.

Examples

1. $x^2\partial/\partial x + y(nx - (n+1)y)\partial/\partial y$, $n \in \mathbb{Z}$, $n \geq 0$,
2. $x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$
3. $x(x - 3y)\partial/\partial x + y(y - 3x)\partial/\partial y$
4. $x(2x - 5y)\partial/\partial x + y(y - 4x)\partial/\partial y$





Uniformizable affine structures (w/singularities on compact curves)

Are:

- Elliptic curves $\mathbb{C}/\Lambda$ and their quotients,
- $\mathbb{CP}^1$ and the quotients by $z \mapsto \omega z$, $\omega^n = 1$,
- $\mathbb{CP}^1$ with the vector field $z\partial/\partial z$, and its quotient under $z \mapsto 1/z$.
- Elliptic curves, quotients of $\mathbb{C}^*$ by groups within $\{z \mapsto \alpha z\}$