Complete holomorphic vector fields and their singular points

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Part I

In dimension 1

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Proposition

Let C be a curve, X a complete vector field on C, $p \in C$, $X(p) = 0$. In suitable coordinates around p either

- $X = \lambda z \partial/\partial z$, $\lambda \in \mathbb{C}$; *or*
- $X = z^2 \partial/\partial z$.

Proof

Facts: normal forms for vector fields

In dimension 1, a vector field is locally equivalent to either:

$$\lambda z\frac{\partial}{\partial z}\ (\lambda\in\mathbb{C}),\ \text{or}\quad \frac{z^{p+1}}{1+\alpha z^p}\frac{\partial}{\partial z}\ (p\geq 1,\alpha\in\mathbb{C}).$$

Second case, $\alpha=1$, the time form is $dT=\left(\frac{1}{z^{p+1}}+\frac{1}{z}\right)dz$.

$z\xrightarrow{f}\exp\left(\int_{z_0}^{z}dT\right)$ maps X to $z\frac{\partial}{\partial z}$. It is one to one if X is semicomplete and in that case it must extend to 0 and conjugate the vector field to $z\partial/\partial z$!

Proof (cont.)

Case $\alpha = 0$. The vector field $z^{p+1}\dfrac{\partial}{\partial z}$ is not semicomplete if $p > 1$:
The function $z \mapsto z^p = w$ maps it to $(p+1)w^2\dfrac{\partial}{\partial w}$.

The catch

Germs of degenerate semicomplete vector fields are very rare!

Part II

Semicomplete vector fields in dimension two

The saddle-nodes

Theorem (Rebelo)

Let X be a germ of semicomplete vector field defined on $(\mathbb{C}^2, 0)$. Suppose that X has at 0 an isolated equilibrium point whose linear part has one vanishing and one non-vanishing eigenvalue. Up to multiplication by a non-vanishing holomorphic function, X is

$$x(1+\lambda y)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},$$

for some $\lambda \in \mathbb{Z}$

Semicomplete vector fields are not too degenerate

Theorem (Rebelo)

X semicomplete vector field on $(\mathbb{C}^2, 0)$, 0 isolated zero of X. If

$$X = X^{(j)} + X^{(j+1)} + \cdots ,$$

then $j \leq 2$.

On Stein surfaces

Theorem (Rebelo, 2007)

Let M be a Stein surface, X a complete holomorphic vector field on M, $p \in M$ isolated zero of X. Then X is non-degenerate at p.

Theorem (Ghys-Rebelo)

*Let X be a germ of semicomplete holomorphic vector field defined on $(\mathbb{C}^2, 0)$. Suppose that X has at 0 an isolated equilibrium point and that the **first jet of X at 0 vanishes**. Then, up to multiplication by a non-vanishing holomorphic function, X is one of the following:*

1. $x^2\partial/\partial x + y(nx - (n+1)y)\partial/\partial y$, $n \in \mathbb{Z}$, $n \geq 0$,
2. $x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$
3. $x(x - 3y)\partial/\partial x + y(y - 3x)\partial/\partial y$
4. $x(2x - 5y)\partial/\partial x + y(y - 4x)\partial/\partial y$

A glimpse of Painlevé's "α method"

Proposition (Rebelo)

Let X semicomplete vector field defined in a ball B around in $\mathbb{C}^2$ with trivial linear part and non-trivial second jet, $X = X^{(2)} + \cdots$.
Then $X^{(2)}$ gives a semicomplete vector field in $\mathbb{C}^2$.

Proof.

Let $\alpha \in \mathbb{C}^*$, $h : B \to \mathbb{C}^2$, $(x, y) \xrightarrow{h} (\alpha^{-1}x, \alpha^{-1}y)$

$$\frac{1}{\alpha} h_* X = X^{(2)} + \alpha X^{(3)} + \alpha^2 X^{(4)} + \cdots +$$

Thus, for every $\alpha \in \mathbb{C}^*$, we have a semicomplete vector field in $\alpha^{-1}B$.
As $\alpha \to 0$, it converges uniformly on compact subsets to $X^{(2)}$ in $\mathbb{C}^2$. □

References

The holomorphic local normal forms for semicomplete holomorphic vector fields on surfaces appear in

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