Complete holomorphic vector fields and their singular points

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Problem

- *Understand holomorphic flows on complex manifolds*
- *Understand holomorphic actions of complex Lie groups on complex manifolds/analytic spaces*

Flows, actions of $\mathbb{C}$, complete vector fields

Complex flow, holomorphic action of $\mathbb{C}$

M complex manifold, $\Phi : \mathbb{C} \times M \to M$

- $\Phi(0, p) = p$
- $\Phi(s, \Phi(t, p)) = \Phi(s + t, p)$

Action $\longrightarrow$ complete vector field: for fixed $p \in M$

$$X(p) = \left. \frac{d}{dt}\Phi(t, p) \right|_{t=0}$$

From vector fields to flows

M manifold, X vector field on M
Does there exist a flow inducing X?

- yes, if M is compact
- in general, no

Rebelo: there are local obstructions!

For instance:

- Understand (semi)complete polynomial vector fields on affine surfaces (Brunella, G.-Rebelo)
- Complete the classification of compact complex surfaces admitting vector fields (Dloussky-Oeljeklaus-Toma)
- Understand compact complex threefolds with a holomorphic action of $\mathrm{SL}(2,\mathbb{C})$ (G.)
- Understand complex surfaces with a holomorphic action of $\mathbb{C}^2$ (Rebelo-Reis-Ferreira)

A result:

Theorem (Rebelo, 2007; G., 2014)

Let V be a two-dimensional normal irreducible Stein space, X a complete holomorphic vector field on V, $p \in V$ isolated zero of X. Either:

- *p is a regular point of V, where X is non-degenerate;*
- *p is a weighted homogeneous singularity (X generates the weighted homotheties); or*
- *p is a cyclic quotient (Hirzebruch-Jung) singularity (X is the quotient of a non-degenerate vector field).*

Locally. . .

M manifold, X vector field on M.
In a chart:

$$X = \sum_{i=1}^{n} f_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i};$$

its integration is equivalent to the system

$$z_i'(t) = f_i(z_1(t)), \ldots, z_n(t)), \ i = 1, \ldots, n$$

Cauchy

For every $p \in M$

- there exists $U \subset \mathbb{C}$ and $\phi : U \to M$, $\phi(0) = p$ solution to the equation;
- the germ of U at 0 is unique.

Vector fields on curves

L complex curve

Three equivalent data:

- X holomorphic vector field without zeros
- ω holomorphic form without zeros (*time form*), $\omega(X) \equiv 1$
- translation structure on L:
  - if $\phi : U \to L$ is a solution of X, ϕ^{-1} is a chart of the translation structure (if $\phi(t)$ is a solution, $\phi(t+c)$ is a solution as well);
  - $D(z) = \int_{z_0}^{z} \omega$ is a chart of the translation structure.

Semicomplete/univalent vector fields

Let L be a curve, X a nowhere vanishing holomorphic vector field on L. The following are equivalent:

1. For every solution $\phi : U \to L$ ($U \subset \mathbb{C}$, $0 \in U$) of X and every pair of paths $\gamma_i : [0,1] \to \mathbb{C}$ ($\gamma_i(0) = 0$, $\gamma_1(1) = \gamma_2(1)$), such that the germ of ϕ at 0 admits analytic continuations along γ_1 and γ_2, both analytic continuations define the same germ at $\gamma_i(1)$.
2. There exists $\Omega \subset \mathbb{C}$ and $\phi : \Omega \to L$ a solution to X such that $\widehat{\phi} : \Omega \to \mathbb{C} \times L$, $\widehat{\phi}(t) = (t, \phi(t))$ is proper.
3. There exists $\Omega \subset \mathbb{C}$ and a covering map $\phi : \Omega \to L$ that is a map between curves with translation structures.
4. For every path $\gamma : [0,1] \to L$ such that $\gamma(0) \neq \gamma(1)$, $\int_\gamma dT \neq 0$.

If X satisfies all (any) of these properties, X is *univalent* or *semicomplete.*

Semicomplete vector fields (Palais, Rebelo, . . .)

- M manifold
- X holomorphic vector field on M
- $\mathcal{F}$ the induced foliation (w/singularities)

Semicompleteness:

X is semicomplete on M if for every leaf L of $\mathcal{F}$ in $M \setminus \mathrm{sing}(X)$, $X|_L$ is semicomplete.

The semiglobal flow of a semicomplete vector field

M manifold, X vector field on M.

If X is semicomplete...

...there exists $\Omega \subset \mathbb{C} \times M$, $\{0\} \times M \subset \Omega$, $\Phi : \Omega \to M$ such that

- $\Phi(0, p) = p$;
- if (s, p), $(t, \Phi(s, p))$ and $(t + s, p)$ are in Ω,

$$\Phi(t + s, p) = \Phi(t, \Phi(s, p));$$

- for $\Omega_p = \{t \mid (t, p) \in \Omega\}$, $\Omega_p \xrightarrow{f} \mathbb{C} \times M$, $(t, p) \mapsto (t, \Phi(t))$ is proper;
- for every $p \in M$,

$$\left.\frac{d}{dt}\Phi(t, p)\right|_{t=0} = X(p).$$

The semiglobal flow of a semicomplete vector field

Reciprocally. . .

If there exists (Ω, Φ) as before, X is semicomplete.

Semicompleteness: some properties

- Complete vector fields are semicomplete.
- The restriction of a semicomplete vector field to an open subset remains semicomplete.
- Semicomplete vector fields may be germified! (Rebelo).
- On a given manifold, the space of semicomplete vector fields is closed (Poincaré, ..., Ghys-Rebelo).

References

Palais's definition of *univalence* (in the context of Lie algebras of vector fields on manifolds) and Rebelo's notion of *semicompleteness* appear in:

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References

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