Complete holomorphic vector fields and their singular points

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The birational point of view: normal forms, combinatorics, …

**Theorem (G.-Rebelo)**

Let $M$ be a surface, $X$ be a reduced holomorphic semicomplete vector field on $M$. Let $p$ be a point where $X(p) = 0$. Either

- $p$ is an isolated non-degenerate singularity of $X$ or,

up to multiplication by a non-vanishing function, $X$ is either:

- $x(1 + \lambda y)\partial/\partial x + y^2\partial/\partial y$, $\lambda \in \mathbb{Z}$,
- $x^p y^q(mx\partial/\partial x - ny\partial/\partial y)$, $pm - qn = 1$ $(m, n \in \mathbb{Z}, m, n \geq 0)$
- $(x^n y^m)^r[(mx + \cdots)\partial/\partial x - (ny + \cdots)\partial/\partial y]$, $(m, n) = 1$, $r \geq 1$
- $x^r y^q\partial/\partial x$ with $r \in \{0, 1, 2\}$, $q \geq 0$. 

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Understandable situations

- Singular points in surfaces with vector fields without separatrices\(^1\)
- Singular points in Stein surfaces\(^1\)
- Vector fields on compact complex surfaces\(^1\)
- Polynomial vector fields on affine surfaces\(^2\) (meromorphic, w/Rebelo)


Theorem (Rebelo)
Let $X$ be a germ of semicomplete holomorphic vector field defined on $(\mathbb{C}^2, 0)$. Then the second jet of $X$ at $0$ does not vanish.

Theorem (Ghys-Rebelo)
Let $X$ be a germ of semicomplete holomorphic vector field defined on $(\mathbb{C}^2, 0)$. Suppose that $X$ has at $0$ an isolated equilibrium point and that the first jet of $X$ at $0$ vanishes. Then, up to multiplication by a non-vanishing holomorphic function, $X$ is one of the following:

- $x^2 \partial/\partial x - y(nx - (n + 1)y)\partial/\partial y$, $n \in \mathbb{Z}$, $n \geq 0$,
- $x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$
- $x(x - 3y)\partial/\partial x + y(y - 3x)\partial/\partial y$
- $x(2x - 5y)\partial/\partial x + y(y - 4x)\partial/\partial y$
A question

Problem

Find analogues of Ghys and Rebelo’s theorem for quasihomogeneous singularities.

ADE singularities (rational double points, Du Val singularities) are likely to play a special role.
The $E_8$ singularity and a quasihomogeneous vector

The $E_8$ singularity

$$V = \{ x^2 + y^3 + z^5 = 0 \}$$

is quasihomogeneous (weights 15, 10, 6). The vector field

$$3iy^2 \frac{\partial}{\partial x} - 2ix \frac{\partial}{\partial y}$$

is tangent to $V$ and quasihomogeneous. It has the first integral $z$.

It is semicomplete. Its general solution is given by

$$(x, y, z) = \left( \frac{i}{2} \varphi'(t), \varphi(t), \sqrt[5]{-\frac{g_3}{4}} \right)$$

for $(\varphi')^2 = 4\varphi^3 - g_3$.

It has a separatrix $s \mapsto (s^3, -s^2, 0)$ in restriction to which the vector field is $is^2 \partial / \partial s$. 
The (complex!) Lorenz system

Lorenz vector field on \( \mathbb{C}^3 \):

\[
\sigma(x - y) \frac{\partial}{\partial x} + (x(\rho - z) - y) \frac{\partial}{\partial y} + (xy - \beta z) \frac{\partial}{\partial z}
\]

Question: which parameters make it semicomplete? \(^3\) (There are some cases! answers by physicists, difficult to track...)


It seems to answer my question. Can we learn something from it towards the understanding of the saddle-node?

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The (complex!) Lorenz system

When compactifying into $\mathbb{CP}^3$, in the affine chart

$$[x : y : z : 1] = [1 : Y : Z : W],$$

the Lorenz vector field reads

$$\frac{1}{W} \left[ (-Z + (\rho - \sigma - 1)W + Y^2W) \frac{\partial}{\partial Y} \right.$$

$$+ (Y - (\beta + \sigma)ZW + \sigma YZW) \frac{\partial}{\partial Z} - \sigma W^2(1 - Y) \frac{\partial}{\partial W} \left. \right]$$

The foliation has a saddle node, the divisor of poles is the strong invariant manifold. When is the germ at $(0, 0, 0)$ semicomplete? See Canille Martins$^4$, Reis$^5$, …

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Ramanujan

The functions $P$, $Q$ and $R$ satisfy the system

$$X = (P^2 - Q) \frac{\partial}{\partial P} + 4(PQ - R) \frac{\partial}{\partial Q} + 6(PR - Q^2) \frac{\partial}{\partial R}$$

$Q$ and $R$ are modular forms, defined in $\mathbf{H} = \{ \Im(z) > 0 \}$ and

$$\frac{Q^3}{R^2} \left( \frac{at + b}{ct + d} \right) = \frac{Q^3}{R^2}(t), \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z})$$

There is a natural boundary. For

$$L = P \frac{\partial}{\partial P} + 2Q \frac{\partial}{\partial Q} + 3R \frac{\partial}{\partial R}, \quad C = \frac{\partial}{\partial P},$$

$$[C, X] = 2L, \quad [L, X] = X, \quad [L, C] = -C$$
Let $X$ be a semicomplete polynomial vector field on $\mathbb{C}^3$. Suppose that it has a solution with an essential boundary (or an uncountable number of singularities).

Is $X$ part of a Lie algebra of rational vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{C})$?
Question: behaviour in dimension three

Let $X$ be a semicomplete polynomial vector field on $\mathbb{C}^3$. Suppose that no fibration is preserved.

Is $X$ part of a Lie algebra of rational vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{C})$?
Semicompleteness and completability

Let $X$ be a semicomplete vector field on $\mathbb{C}^n$. Can it be completed? (Does there exist a manifold $M$, $\mathbb{C}^n \hookrightarrow M$ and a complete vector field on $M$ that extends $X$?)

Let $X$ be a germ semicomplete vector field on $(\mathbb{C}^n, 0)$. Is it the local model of a complete vector field? (Does there exist a manifold $M$ and a complete vector field on $M$ whose germ at some point is equivalent to $X$?)

Palais$^6$ proved that such a manifold exists in the realm of non-Hausdorff manifolds by giving a universal construction. Another way to formulate these questions would be: does Palais’s construction give, in these cases, a Hausdorff manifold?

Some questions

- Let $X$ be a germ of semicomplete vector field on $(\mathbb{C}^3, 0)$ with an isolated singularity. Does it have a separatrix? Can its second jet be trivial? What if the vector field preserves a volume form?