Complete holomorphic vector fields and their singular points

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The birational point of view: normal forms, combinatorics, ...

Theorem (G.-Rebelo)

Let M be a surface, X be a reduced holomorphic semicomplete vector field on M. Let p be a point where $X(p) = 0$. Either

- *p is an isolated non-degenerate singularity of X or,*

up to multiplication by a non-vanishing function, X is either:

- $x(1+\lambda y)\partial/\partial x + y^2\partial/\partial y$, $\lambda \in \mathbf{Z}$,
- $x^p y^q(mx\partial/\partial x - ny\partial/\partial y)$, $pm - qn = 1$ $(m, n \in \mathbb{Z},\ m, n \geq 0)$
- $(x^n y^m)^r[(mx + \cdots)\partial/\partial x - (ny + \cdots)\partial/\partial y]$, $(m, n) = 1$, $r \geq 1$
- $x^r y^q \partial/\partial x$ *with* $r \in \{0, 1, 2\}$, $q \geq 0$.

Understandable situations

- Singular points in surfaces with vector fields without separatrices[1]
- Singular points in Stein surfaces[1]
- Vector fields on compact complex surfaces[1]
- Polynomial vector fields on affine surfaces[2] (meromorphic, w/Rebelo)

[1] Guillot, A. Vector fields, separatrices and Kato surfaces. *Ann. Inst. Fourier (Grenoble)* 64 (2014), no. 3, 1331–1361.

[2] Guillot, Rebelo, Semicomplete meromorphic vector fields on complex surfaces. *J. Reine Angew. Math.* 667 (2012), 27–65.

Theorem (Rebelo)

Let X be a germ of semicomplete holomorphic vector field defined on $(\mathbf{C}^2, 0)$. Then the second jet of X at 0 does not vanish.

Theorem (Ghys-Rebelo)

Let X be a germ of semicomplete holomorphic vector field defined on $(\mathbf{C}^2, 0)$. Suppose that X has at 0 an isolated equilibrium point and that the first jet of X at 0 vanishes. Then, up to multiplication by a non-vanishing holomorphic function, X is one of the following:

- $x^2\partial/\partial x - y(nx - (n+1)y)\partial/\partial y$, $n \in \mathbb{Z}$, $n \geq 0$,
- $x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$
- $x(x - 3y)\partial/\partial x + y(y - 3x)\partial/\partial y$
- $x(2x - 5y)\partial/\partial x + y(y - 4x)\partial/\partial y$

A question

Problem

Find analogues of Ghys and Rebelo's theorem for quasihomogeneous singularities.

ADE singularities (rational double points, Du Val singularities) are likely to play a special role.

The E_8 singularity and a quasihomogeneous vector

The E_8 singularity

$$V = \{x^2 + y^3 + z^5 = 0\}$$

is quasihomogeneous (weights 15, 10, 6). The vector field

$$3iy^2 \frac{\partial}{\partial x} - 2ix \frac{\partial}{\partial y}$$

is tangent to V and quasihomogeneous. It has the first integral z. It is semicomplete. Its general solution is given by

$$(x, y, z) = \left(\frac{i}{2} \wp'(t), \wp(t), \sqrt[5]{-\frac{g_3}{4}} \right)$$

for $(\wp')^2 = 4\wp^3 - g_3$.
It has a separatrix $s \mapsto (s^3, -s^2, 0)$ in restriction to which the vector field is $is^2\partial/\partial s$.

The (complex!) Lorenz system

Lorenz vector field on $\mathbf{C}^3$:

$$\sigma(x-y)\frac{\partial}{\partial x} + (x(\rho - z) - y)\frac{\partial}{\partial y} + (xy - \beta z)\frac{\partial}{\partial z}$$

Question: which parameters make it semicomplete?[3] (There are some cases! answers by physicists, difficult to track...)

[3] While looking for references I found the 2015 article:
Nikolay A. Kudryashov, Analytical Solutions of the Lorenz System, *Regul. Chaotic Dyn.* 20 (2015), no. 2, pp. 123–133.
It seems to answer my question. Can we learn something from it towards the understanding of the saddle-node?

The (complex!) Lorenz system

When compactifying into $\mathbf{CP}^3$, in the affine chart

$$[x : y : z : 1] = [1 : Y : Z : W],$$

the Lorenz vector field reads

$$\frac{1}{W}\left[(-Z + (\rho - \sigma - 1)W + Y^2W)\frac{\partial}{\partial Y} + (Y - (\beta + \sigma)ZW + \sigma YZW)\frac{\partial}{\partial Z} - \sigma W^2(1 - Y)\frac{\partial}{\partial W}\right]$$

The foliation has a saddle node, the divisor of poles is the strong invariant manifold. When is the germ at $(0, 0, 0)$ semicomplete? See Canille Martins[4], Reis[5], ...

[4] J. C. Canille Martins, Contribuição ao estudo local de fluxos holomorfos com ressonância, Ph.D. Thesis, IMPA-CNPq, Brasil (1985)

[5] Reis, Helena, Semi-complete vector fields of saddle-node type in $\mathbb{C}^n$. *Trans. Amer. Math. Soc.* 360 (2008), no. 12, 6611–6630.

Essential boundaries

Ramanujan

The functions P, Q and R satisfy the system

$$X = (P^2 - Q)\frac{\partial}{\partial P} + 4(PQ - R)\frac{\partial}{\partial Q} + 6(PR - Q^2)\frac{\partial}{\partial R}$$

Q and R are modular forms, defined in $\mathbf{H} = \{\Im(z) > 0\}$ and

$$\frac{Q^3}{R^2}\left(\frac{at+b}{ct+d}\right) = \frac{Q^3}{R^2}(t), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$$

There is a natural boundary. For

$$L = P\frac{\partial}{\partial P} + 2Q\frac{\partial}{\partial Q} + 3R\frac{\partial}{\partial R},\ C = \frac{\partial}{\partial P},$$

$$[C, X] = 2L,\ [L, X] = X,\ [L, C] = -C$$

Question: mechanisms that produce essential boundaries

Let X be a semicomplete polynomial vector field on $\mathbf{C}^3$. Suppose that it has a solution with an essential boundary (or an uncountable number of singularities).

Is X part of a Lie algebra of rational vector fields isomorphic to $\mathfrak{sl}(2, \mathbf{C})$?

Question: behaviour in dimension three

Let X be a semicomplete polynomial vector field on $\mathbf{C}^3$. Suppose that no fibration is preserved.

Is X part of a Lie algebra of rational vector fields isomorphic to $\mathfrak{sl}(2, \mathbf{C})$?

Semicompleteness and completability

Let X be a semicomplete vector field on $\mathbf{C}^n$. Can it be completed? (Does there exist a manifold M, $\mathbf{C}^n \hookrightarrow M$ and a complete vector field on M that extends X?)

Let X be a germ semicomplete vector field on $(\mathbf{C}^n, 0)$. Is it the local model of a complete vector field? (Does there exist a manifold M and a complete vector field on M whose germ at some point is equivalent to X?)

Palais[6] proved that such a manifold exists in the realm of non-Hausdorff manifolds by giving a universal construction. Another way to formulate these questions would be: does Palais's construction give, in these cases, a Hausdorff manifold?

[6] Palais, Richard S. A global formulation of the Lie theory of transformation groups. *Mem. Amer. Math. Soc.* 22 (1957)

Some questions

- Let X be a germ of semicomplete vector field on $(\mathbf{C}^3, 0)$ with an isolated singularity. Does it have a separatrix? Can its second jet be trivial? What if the vector field preserves a volume form?