MMP for foliations of rank one (on 3folds)

$/\mathbb{C}$

X = normal projective variety of dim n

$T_{\mathcal{F}} \hookrightarrow T_X$ subsheaf of rank one

In the smooth locus of $\mathcal{F}$ & X

the foliation is defined by a section of T_X

We will assume that $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier.

if $K_{\mathcal{F}}$ is Cartier then $\mathcal{F}$ is defined by ∂.

Ex. let $x \in X$ smooth pt

s.t. $x \in \operatorname{Sing} \mathcal{F}$ isolated pt.

Then if $m \subset \mathcal{O}_X$ ideal of x

then $\partial(m) \subset m$

Leibnitz $\Rightarrow$ $\partial(m^k) \subset m^k \quad \forall k \geq 1$

$$\rightsquigarrow \partial_k : \mathfrak{m}/\mathfrak{m}^{k+1} \longrightarrow \mathfrak{m}/\mathfrak{m}^{k+1}$$

Jordan decomposition

$$\partial_k = \underset{\substack{\nearrow \\ \text{semisimple}}}{\partial_{S,k}} + \underset{\substack{\nwarrow \\ \text{nilpotent.}}}{\partial_{N,k}} \qquad \forall k.$$

<u>Theorem</u> (McQuillan - Panazzolo)

$\mathcal{F}$ is log canonical at $x \iff \partial_S \neq 0$

<u>Idea</u> Find $x_1, \dots, x_n$ around x

$\lambda_1, \dots, \lambda_n \in \mathbb{C}$ s.t.

$$\partial_S = \sum \lambda_i x_i \partial_{x_i}$$

<u>Theorem</u> (M.P. 2013)

$\mathcal{F}$ rank one foliation on $X = 3$-fold.

Then $\exists$ sequence of weighted blow up

$X_n = Y \to X_{n-1} \to \dots \to X_1 = X$

so that $\mathcal{F}_{X_n}$ has canonical singularities

Note that X_n has quotient singularities

and either $\mathcal{F}_{X_n}$ is terminal

or X_n has $\mathbb{Z}/_2$-quot. sing. $\Big\}$ admissible singularities.

$\boxed{n \geqslant 2}$ Rank one

Bogomolov - McQuillen if X is smooth & $\mathcal{F}$ is terminal then $\mathcal{F}$ is smooth.

B. McQ. Bend and break & Cone theorem hold

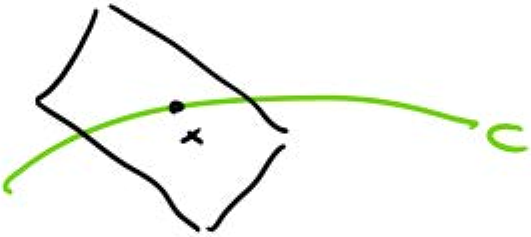
Theorem : X = smooth proj. variety

$\mathcal{F}$ = foliation on X

$C \subset X$ curve s.t.

$T_{\mathcal{F}}|_C$ is ample v.b.

Then $\forall x \in C$ general, the leaf is algebraic and rationally connected



In particular, if $\mathrm{rk}\, \mathcal{F} = 1$ then

$T_{\mathcal{F}}|_C$ is ample $\iff K_{\mathcal{F}} . C < 0$

Thus Bend & Break holds.

<u>Theorem</u> (McQuillan)

If $\mathcal{F}$ foliation of rank one on X with quotient singularities and $\mathcal{F}$ is canonical and $K_{\mathcal{F}}$ is pseudo-effective then $\exists$ $X \dashrightarrow Y$ birational contraction s.t. $K_{\mathcal{F}_Y}$ is nef

Theorem (C - Spicer):

If $\mathcal{F}$ foliation of rank one with admissible sing. on a 3-fold X, such that $K_{\mathcal{F}}$ is ps. eff.

Then

$\exists \; X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots\cdots \; X_n = Y$ flips/div contr.

s.t.

$K_{\mathcal{F}_{X_n}}$ is nef.

Proposition: Let $X \dashrightarrow X'$ is a step of the MMP for $\mathcal{F}$ = foliation on X.

if $\mathcal{F}$ has admissible singularities then also $\mathcal{F}'$ on X' has admissible singularities.

Idea of the proof of main theorem:

Assume $K_{\mathcal{F}}$ is not nef.

by the cone theorem, $\exists\, R = \mathbb{R}_+[\overline{C}]$ extremal ray of $\overline{NE}(X)$ which is $K_{\mathcal{F}}$-negative

$$K_{\mathcal{F}} \cdot \overline{C} < 0$$

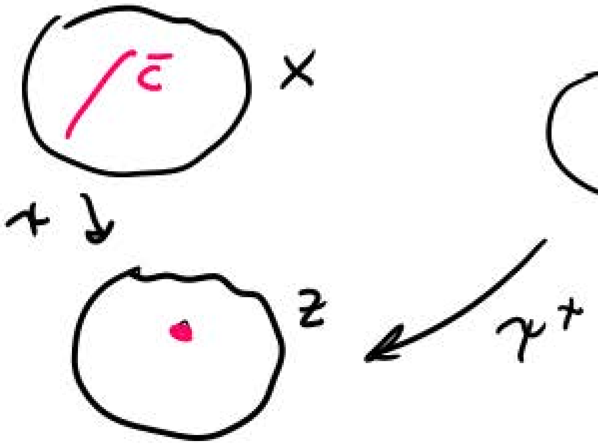
Fact $\text{locus}(R) = \bigcup_{C \in R} C$ is Zariski closed

$$\begin{tikzcd}
\text{BDPP} \Rightarrow \text{locus}(R) \neq X \arrow[r] \arrow[dr] & \boxed{\text{curve}} \\
& \text{surface}
\end{tikzcd}$$

Assume it is a curve

$\exists\, \psi : X \to Z$ birational s.t. $\text{Exc}\,\psi = \text{locus}(R)$

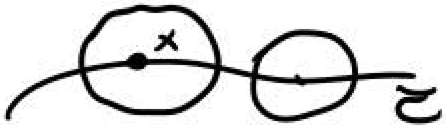
and Z = alg. space.



Idea We want to find Δ s.t. $(K_X + \Delta) \cdot \overline{C} < 0$ and (X, Δ) is log canonical

As in the case of rank 2 foliations it is enough to construct such Δ on a formal neighbourhood of $\overline{C}$

$\exists!\ x \in \overline{C}$ on which $\mathcal{F}$ is not terminal

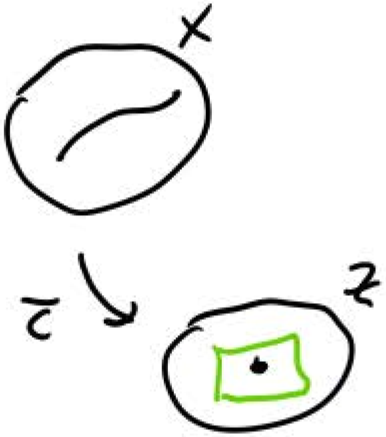


x sing pt $\Longrightarrow$ around x we can write

$$\partial = \partial_S + \partial_N$$

$$\partial_S = \sum \lambda_i x_i \partial_{x_i}$$

$\{x_i = 0\}$ is invariant.



$\mathcal{F}_Z$ = foliation on Z

$z = \psi(\overline{C})$

but $K_{\mathcal{F}_Z}$ is not $\mathbb{Q}$-Cartier

Ingredient ① $\exists\, \overset{\text{surface}}{D}$ on X s.t.

$K_{\mathcal{F}} + D$ is $\mathbb{Q}$-Cartier and $\exists\, m > 0$

s.t. $m(K_{\mathcal{F}} + D) \sim 0$ in a neighbourhood of $\overline{C}$

$\Rightarrow$ if $D_Z = \psi(D)$ then

$K_{\mathcal{F}_Z} + D_Z$ is $\mathbb{Q}$ Cartier on Z.

② $\exists\, S_1, \ldots, S_k$ invariants divisors containing z

③ $\rightsquigarrow$ S_i induces $\overline{S}_i$ on X which are $\mathcal{F}$-invariant.

We can $\Delta = \sum \overline{S}_i$ & $(K_X + \Delta).\overline{C} < 0$.

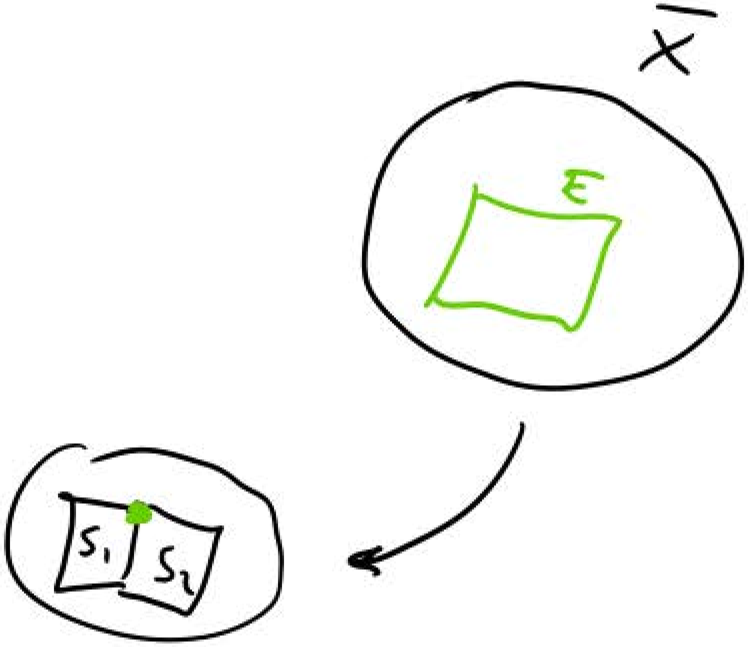
④ We need to show that (X, Δ) is log canonical.

Remark if $S_1, \dots, S_n$ are $\mathcal{F}$-invariant then in general $(X, \sum S_i)$ is not log canonical

But ok for closed pts:

Easy case $\exists\ \mu : \overline{X} \to X$ log resolution for $(X, \sum S_i)$

s.t. $\mathrm{Exc}\,\mu = E$ irr. divisor
and $\mu(E) = pt$



We have

$$K_{\bar{Y}} = \mu^* K_Y$$

$$K_{\bar{X}} + E = \mu^*(K_X) + aE \qquad \underline{\text{Goal}}\ a \geq 0$$

Use adjunction both for $\bar{Y}$ and for $(\bar{X}, E)$

$$\begin{array}{l} (K_{\bar{X}} + E)|_E = a\, E|_E \\ \quad \parallel \\ \quad K_E \end{array}$$

$-E|_E$ is ample

$$K_{\bar{Y}|_E} = 0$$

$K_{\mathcal{G}} + \Theta$ $\Theta \geq 0$ on E

Assume

$\Theta \neq 0 \Rightarrow K_{\mathcal{G}} = -\Theta \Rightarrow \mathcal{G}$ is uniruled

$\Rightarrow E$ is covered by curves Σ s.t.

$$\left.\begin{array}{l} K_E \cdot \Sigma < 0 \\ -E_{|E} \cdot \Sigma > 0 \end{array}\right\} \Rightarrow a > 0 \Rightarrow \text{OK!}$$

$\square$