Fano Foliations 0 - Algebraicity of smooth formal schemes and applications to foliations

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Plan for the mini-course

- Lecture 0: Algebraicity of smooth formal schemes and applications to foliations
- Lecture 1: Definition, examples and first properties (by C. Araujo)
- Lecture 2: Adjunction formula and applications
- Lecture 3: Classification of Fano foliations of large index (by C. Araujo)

Algebraicity of (smooth) formal schemes - Setup

X (projective) variety over a field K
$x \in X(K)$

$\widehat{X}$ formal completion of X at x
$\widehat{V} \subseteq \widehat{X}$ a formal subscheme, *i.e.*, an increasing sequence of closed subschemes of X with support $\{x\}$

$$\{x\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_i \subseteq \cdots \subseteq X$$

and $\widehat{V} = \varinjlim V_i$ (in the category of ringed spaces over K).

The dimension of $\widehat{V}$ is the Krull dimension of the K-algebra $\varprojlim H^0(X, \mathscr{O}_{V_i})$ of regular functions on $\widehat{V}$.

Say that $\widehat{V}$ is smooth if the K-algebra $\varprojlim H^0(X, \mathscr{O}_{V_i})$ is isomorphic to $K[[t_1, \ldots, t_N]]$ with $N = \dim \widehat{V}$. *Then we may assume without loss of generality that* $H^0(X, \mathscr{O}_{V_i})$ *identifies with* $K[[t_1, \ldots, t_N]]/(t_1, \ldots, t_N)^{i+1}$.

Example

$K = \mathbb{C}$
$V^{an} \subseteq (X^{an}, x)$ germ of smooth complex analytic variety at $x \rightsquigarrow$ smooth formal subscheme $\widehat{V}$ of X through x given by $\mathscr{O}_{V_i} := \mathscr{O}_{V^{an},x}/\mathscr{M}_x^{i+1}$ ($\dim \widehat{V} = \dim V^{an}$).

Lemma

TFAE

1. *There exists a closed subvariety V of X over K with $x \in V$ and such that $\widehat{V}$ is a branch of V through x (a component of the formal completion of V at x).*
2. *The dimension of the Zariski closure of $\widehat{V}$ in X is* $\dim \widehat{V}$.

Example (The nodal plane cubic curve)

The formal completion of the ring $K[x,y]/(y^2 - x^2 - x^3)$ at the origin (x,y) is isomorphic to $K[[u,v]]/(uv)$.

Algebraicity of (smooth) formal schemes - A definition 2

Proof.

Let Y be the Zariski closure of $\widehat{V}$ in X. Then Y is an integral subscheme of X passing through x. Moreover, any branch of Y through x has dimension $\dim Y$.

(1) $\implies$ (2)

Then we must have $\widehat{V} \subseteq Y \subseteq V$. Moreover, $\widehat{V}$ is a branch of Y at x. Hence $\dim Y = \dim \widehat{V}$.

(2) $\implies$ (1)

Since $\dim Y = \dim \widehat{V}$ by assumption and $\widehat{V} \subseteq Y$, $\widehat{V}$ is a branch of $Y =: V$ through x. □

Definition

Say that $\widehat{V}$ is algebraic if the equivalent conditions below are satisfied.

1. There exists a closed subvariety V of X over K with $x \in V$ and such that $\widehat{V}$ is a branch of V through x (a component of the formal completion of V at x).
2. The dimension of the Zariski closure of $\widehat{V}$ in X is $\dim \widehat{V}$.

Example

$f \in K[[t_1, \dots, t_N]]$ with $f(0) = 0 \rightsquigarrow$

$$\widehat{V} := \mathrm{Graph}(f) \subset \mathbb{A}_K^{N+1} \quad \text{through} \quad 0$$

formally defined by the principal ideal generated by $z - f(t_1, \dots, t_N)$ in $K[[t_1, \dots, t_N, z)]]$.

Write $f = f_{\leqslant i} \bmod (t_1, \dots, t_N, z)^{i+1}$ with $f_{\leqslant i} \in K[t_1, \dots, t_N]$. Let $V_i \subset \mathbb{A}_K^{N+1}$ be the closed subscheme of $\mathrm{Spec}\, K[t_1, \dots, t_N]/(t_1, \dots, t_N, z)^{i+1} \subset \mathbb{A}_K^{N+1}$ defined by the principal ideal generated by $f_{\leqslant i} \rightsquigarrow \widehat{V} = \varinjlim V_i$.

Then $\widehat{V}$ is algebraic if and only if f is algebraic over $K(t_1, \dots, t_N)$.

Example

$K = \mathbb{C}$, X smooth complex variety, $\mathscr{G} \subset T_X$ regular foliation, L the germ of leaf of $\mathscr{G}$ through a point $x \in X \rightsquigarrow$ a smooth formal subscheme $\widehat{V}$ of X through x.

Then $\widehat{V}$ is algebraic if and only if L is algebraic.

X projective variety over a field K
$x \in X(K)$
$\mathscr{L}$ ample line bundle on X
$\widehat{V} \subset \widehat{X}$ smooth formal subscheme of the formal completion $\widehat{X}$ of X at x

Proposition A (Bost 2004)

The formal scheme $\widehat{V}$ is algebraic if and only if there exists $c > 0$ such that, for any positive integer j and any section $s \in H^0(X, \mathscr{L}^{\otimes j})$ with $s_{|\widehat{V}} \not\equiv 0$, the multiplicity $\mathrm{mult}_x(s_{|\widehat{V}})$ of $s_{|\widehat{V}}$ at x is $\leqslant cj$.

$(\mathrm{mult}_x(s_{|\widehat{V}}) := \max\{i \mid s_{|V_i} \equiv 0\} \in \mathbb{N} \cup \{+\infty\})$

Proof of sufficiency.

Let Y be the Zariski closure of $\widehat{V}$ in X. Set $d := \dim \widehat{V}$.
Since $\widehat{V} \subseteq Y$, we have $d \leqslant \dim Y$.
$\rightsquigarrow$ Need to show that $\dim Y \leqslant d$.

Enough to show that $\dim H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \leqslant C \cdot j^d$ for some $C > 0$ and any integer j large enough.

Fix j_0 (Serre's vanishing theorem) such that the restriction map

$$H^0(X, \mathscr{L}^{\otimes j}) \to H^0(Y, \mathscr{L}_{|Y}^{\otimes j})$$

is surjective for any $j \geqslant j_0$.

For $i \geqslant 0$, set

$$\begin{aligned} F^i H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) &:= \big\{ s \in H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \,|\, \mathrm{mult}_x(s_{|\widehat{V}}) \geqslant i \big\} \\ &= \big\{ s \in H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \,|\, s_{|V_i} \equiv 0 \big\}. \end{aligned}$$

$\rightsquigarrow$ decreasing filtration on $H^0(Y, \mathscr{L}_{|Y}^{\otimes j})$.

Notice that the restriction map $H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \to H^0(Y, \mathscr{L}_{|\widehat{V}}^{\otimes j})$ is injective $\rightsquigarrow$ $\mathrm{mult}_x(s_{|\widehat{V}}) < +\infty$ if $s \in H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \setminus \{0\}$. In other words,

$$\cap_{i \geqslant 0} F^i H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) = \{0\}.$$

The exact sequence

$$0 \to \mathscr{I}_{V_{i+1}/Y} \to \mathscr{I}_{V_i/Y} \to \mathscr{N}^*_{V_i/V_{i+1}} \to 0$$

yields an injective map

$$F^i H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) / F^{i+1} H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \hookrightarrow H^0(V_0, \mathscr{N}^*_{V_i/V_{i+1}} \otimes \mathscr{L}_{|V_0}^{\otimes j}).$$

Moreover $\mathscr{N}_{V_i/V_{i+1}} \otimes \mathscr{L}_{|V_0}^{\otimes j} \cong \mathrm{S}^i \mathscr{N}_{V_0/V_1} \otimes \mathscr{L}_{|V_0}^{\otimes j}$, and hence

$$\dim F^i H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) / F^{i+1} H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \leqslant \binom{d+i-1}{i}.$$

Now, suppose $j \geqslant j_0$. By assumption

$$F^i H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) = \{0\} \quad \textit{if} \quad i > cj.$$

Therefore

$$\begin{aligned}\dim H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) &\leqslant \sum_{i\geqslant 0} \dim F^i H^0(Y, \mathscr{L}_{|Y}^{\otimes j})/F^{i+1} H^0(Y, \mathscr{L}_{|Y}^{\otimes j}) \\ &\leqslant \sum_{0\leqslant i\leqslant [cj]} \binom{d+i-1}{i} \\ &\sim \frac{c^d}{d!} j^d \quad \text{as } j \text{ goes to infinity.}\end{aligned}$$

References

Bost, J.-B.: « Algebraic leaves of algebraic foliations over number fields ». Publ. Math. Inst. Hautes Études Sci. 93, 161-221 (2001).

Bost, J.-B.: « Germs of analytic varieties in algebraic varieties: canonical metrics and arithmetic algebraization theorems ». In: Adolphson, A., Baldassarri, F., Berthelot, P., Katz, N., Loeser, F. (eds.) Geometric Aspects of Dwork Theory, vol. I. Walter de Gruyter II, Berlin (2004).

Proposition B

Let X and Y be complex projective varieties with $Y \subseteq X$, and let $X_1 \subseteq X$ be a dense Zariski open set such that $Y_1 := X_1 \cap Y \subseteq Y_{\mathrm{reg}}$ and such that $Y \setminus Y_1$ has codimension at least 2. Let $\mathscr{L}$ be an ample line bundle on X. Let $V_1^{an} \subset X_1^{an}$ be a germ of smooth locally closed analytic submanifold along Y_1^{an}.

Then V_1^{an} is algebraic if and only if there exists $c > 0$ such that, for any positive integer j and any section $s \in H^0(X, \mathscr{L}^{\otimes j})$ such that $s_{|V_1^{an}} \not\equiv 0$, the multiplicity $\mathrm{mult}_{Y_1}(s_{|V_1^{an}})$ of $s_{|V_1^{an}}$ along Y_1 is $\leqslant cj$.

Remark

A similar statement holds for $\widehat{V}$ a smooth formal scheme over any field K with support Y_1.

The proof is similar to that of Proposition A. Let $Z \subset X$ be the Zariski closure of V_1^{an} in X. We need to show that $\dim Z = \dim V_1^{an} =: d$. Set $Z_1 := Z \cap X_1$.
Let $V_i \subset Z_1$ be the subscheme defined by $\mathscr{I}^{i+1}_{Y_1^{an}/V_1^{an}}$ for $i \geqslant 0$.
For $i \geqslant 0$, set

$$F^i H^0(Z, \mathscr{L}^{\otimes j}_{|Z}) = \big\{ s \in H^0(Z, \mathscr{L}^{\otimes j}_{|Z}) \,|\, s_{|V_i} \equiv 0 \big\}.$$

$\rightsquigarrow$ decreasing filtration on $H^0(Z, \mathscr{L}^{\otimes j}_{|Z})$.
For $j \gg 1$ and $i > cj$, $F^i H^0(Z, \mathscr{L}^{\otimes j}_{|Z}) = \{0\}$.
Thus

$$\dim H^0(Z, \mathscr{L}^{\otimes j}_{|Z}) \leqslant \sum_{0 \leqslant i \leqslant [cj]} \dim F^i H^0(Z, \mathscr{L}^{\otimes j}_{|Z}) / F^{i+1} H^0(Z, \mathscr{L}^{\otimes j}_{|Z}).$$

Suppose for simplicity that $Y_1 = Y$. Then

$$\dim F^i H^0(Z, \mathscr{L}^{\otimes j}_{|Z}) / F^{i+1} H^0(Z, \mathscr{L}^{\otimes j}_{|Z}) \leqslant \dim H^0(Y, S^i \mathscr{N}^*_{Y/V} \otimes \mathscr{L}^{\otimes j}_{|Y}).$$

Asymptotic Riemann-Roch $\rightsquigarrow$ there exists $C > 0$ such that
$\dim H^0(Y, S^i \mathscr{N}^*_{Y/V} \otimes \mathscr{L}^{\otimes j}_{|Y}) \leqslant C(i+j)^{\dim Y + \operatorname{rank} \mathscr{N}_{Y/V} - 1}$.

Application to foliations 1

Theorem (Bost, Campana - Păun, – 2018)

Let X be a normal complex projective variety, let $\mathscr{L}$ be an ample Cartier divisor, and let $\mathscr{G} \subseteq T_X$ be a foliation. Suppose that there exists $c > 0$ such that

$$h^0(X, S^{[i]}\mathscr{G}^* \otimes \mathscr{L}^{\otimes j}) = 0$$

for any positive integer j and any natural number i satisfying $i > cj$.

Then $\mathscr{G}$ has algebraic leaves.

Remark

X smooth and $\mathscr{G}$ regular $\rightsquigarrow$ the condition above can be rephrased as follows: the tautological class on $\mathbb{P}_X(\mathscr{G}^*)$ is not pseudo-effective.

$X_1 \subset X_{\mathrm{reg}}$ open set where $\mathscr{G}_{|X_{\mathrm{reg}}}$ is a subbundle of $T_{X_{\mathrm{reg}}}$,
$Z_1 := X_1 \times X_1$ and $Z := X \times X$,
$\mathscr{L}_Z := \mathscr{L} \boxtimes \mathscr{L}$,
$X_1 =: Y_1 \subset Z_1$ and $X =: Y \subset Z$ diagonal.

$V_1^{an} \subset Z_1^{an}$ germ of the analytic graph of $(X_1, \mathscr{G}_{|X_1})$ along the diagonal Y_1
$\rightsquigarrow Y_1^{an} \subset V_1^{an}$ and $\mathscr{N}_{Y_1^{an}/V_1^{an}} \cong \mathscr{G}_{|X_1^{an}}$,
$\mathscr{G}$ is algebraically integrable if and only if V_1^{an} is algebraic.

Assumption ($X \setminus X_1$ of codimension $\geqslant 2$) $\rightsquigarrow$

$$h^0(Y_1^{an}, S^i \mathscr{N}^*_{Y_1^{an}/V_1^{an}} \otimes \mathscr{L}^{\otimes j}_{|Y_1^{an}}) = 0$$

if $i > cj > 0$
$\rightsquigarrow \mathrm{mult}_{Y_1}(s_{|V_1^{an}}) \leqslant 2cj$ for any positive integer j and any section $s \in H^0\big(Z, \mathscr{L}_Z^{\otimes j}\big)$ such that $s_{|V_1^{an}}$ is non-zero.
(use

$$0 \to \mathscr{I}^{i+1}_{Y_1^{an}/V_1^{an}} \to \mathscr{I}^{i}_{Y_1^{an}/V_1^{an}} \to S^i \mathscr{N}^*_{Y_1^{an}/V_1^{an}} \to 0)$$

$\rightsquigarrow$ The theorem follows from Proposition B applied to $(Z, Z_1, Y, Y_1, \mathscr{L}_Z)$ and V_1^{an}.

Theorem (Campana - Păun 2019)

Let X be a normal $\mathbb{Q}$-factorial complex projective variety, $\alpha \in \mathrm{N}_1(X)_{\mathbb{R}}$ a movable curve class, and $\mathscr{G} \subset T_X$ a foliation on X of positive rank. Suppose that $\mu_\alpha^{\min}(\mathscr{G}) > 0$.

Then $\mathscr{G}$ is algebraically integrable and the closure of a general leaf is rationally connected.

$\alpha \in \mathrm{N}_1(X)_{\mathbb{R}}$ is called movable if $D \cdot \alpha \geqslant 0$ for any effective divisor D.

$\mu_\alpha(\mathscr{Q}) = \frac{\det(\mathscr{Q})\cdot\alpha}{\mathrm{rank}(\mathscr{Q})}$, $\mathscr{Q} \neq 0$ torsion-free.

$\mu_\alpha^{\min}(\mathscr{G}) := \inf\left\{\mu_\alpha(\mathscr{Q}) \,|\, \mathscr{Q} \neq 0 \text{ is a torsion-free quotient of } \mathscr{G}\right\}$.

Algebraicity - Suppose for simplicity that X is smooth.

Fact (Campana - Peternell - Toma 2011): $\mu_\alpha^{\min}(S^i\mathscr{G}) = i\mu_\alpha^{\min}(\mathscr{G})$

$\mathscr{L}$ ample line bundle on X

$h^0(X, S^i\mathscr{G}^* \otimes \mathscr{L}^{\otimes j}) \neq 0 \rightsquigarrow$ there is a non-zero map $S^i\mathscr{G} \to \mathscr{L}^{\otimes j}$

$\rightsquigarrow i\mu_\alpha^{\min}(\mathscr{G}) = \mu_\alpha^{\min}(S^i\mathscr{G}) \leqslant \mu_\alpha(\mathscr{L}^{\otimes j}) = j\mu_\alpha(\mathscr{L}) \leqslant j(\mu_\alpha(\mathscr{L}) + 1)$

Thus

$$h^0(X, S^i\mathscr{G}^* \otimes \mathscr{L}^{\otimes j}) = 0$$

if $i > \frac{1+\mu_\alpha(\mathscr{L})}{\mu_\alpha^{\min}(\mathscr{G})} j$ so that the theorem applies
($\mu_\alpha^{\min}(\mathscr{G}) > 0$ by assumption and $\mu_\alpha(\mathscr{L}) \geqslant 0$ since α is movable).

Theorem (Bost 2001, Bogomolov - McQuillan 2016)

Let X be a normal complex projective variety, and $\mathscr{G}$ a foliation on X. Let $C \subset$ be a complete curve disjoint from the singular loci of X and $\mathscr{G}$. Suppose that the restriction $\mathscr{G}_{|C}$ is an ample vector bundle on C.

Then the leaf of $\mathscr{G}$ through any point of C is an algebraic variety. The closure of a leaf through a general point of C is rationally connected.

(Algebraicity is a consequence of Proposition B)

Thanks!