Fano Foliations 2 - Adjunction formula and applications

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Motivation

Setup 1

X smooth (projective) variety over $\mathbb{C}$
$\mathscr{G}$ regular foliation on X of rank $r > 0$
L (projective) algebraic leaf

$\rightsquigarrow T_L = \mathscr{G}_{|L} \subseteq T_{X\,|L}$ and $K_L \sim_{\mathbb{Z}} K_{\mathscr{G}\,|L}$ (« adjunction formula »).

Setup 2

X smooth (projective) variety over $\mathbb{C}$
$\mathscr{G}$ foliation on X of rank $r > 0$
$L \subset X$ subvariety which is not entirely contained in the singular locus of $\mathscr{G}$ and such that $L \cap X \setminus \operatorname{Sing}\mathscr{G}$ is a leaf of $\mathscr{G}_{|X \setminus \operatorname{Sing}\mathscr{G}}$

$\rightsquigarrow$ We ask whether there is an adjunction formula in this setting.

Let $n\colon F \to L \subseteq X$ be the normalization.

The r-th wedge product of the inclusion $\mathscr{G} \subseteq T_X$ gives rise to a non-zero map $\Omega_X^r \to \mathscr{O}_X(K_{\mathscr{G}})$.

Proposition A (Araujo - D. 2013)

The map $\Omega_X^r \to \mathscr{O}_X(K_{\mathscr{G}})$ induces a non-zero map $\Omega_F^r \to n^*\mathscr{O}_X(K_{\mathscr{G}})$.

In particular, there is a canonically defined effective (integral) Weil divisor Δ_F on F such that

$$K_F + \Delta_F \sim_{\mathbb{Z}} K_{\mathscr{G}}{}_{|F}.$$

$\Omega_F^r \to n^*\mathscr{O}_X(K_{\mathscr{G}})$ induces a non-zero map $\mathscr{O}_F(K_F) \cong \Omega_F^{[r]} \to n^*\mathscr{O}_X(K_{\mathscr{G}})$

Definition

The pair (F, Δ_F) is called a « log leaf ».

There is a factorization

$$\begin{array}{ccc} \Omega^r_{X\,|L} & \longrightarrow & \mathscr{O}_X(K_{\mathscr{G}})_{|L} \\ \downarrow & & \| \\ \Omega^r_L & \longrightarrow & \mathscr{O}_X(K_{\mathscr{G}})_{|L} \end{array}$$

so that the proof of the proposition boils down to the following Seidenberg extension type result.

Proposition B (Araujo - D. - Kovács 2008)

Let Z be a complex variety with normalization $n\colon Z_1 \to Z$. Let $\mathscr{L}$ be a line bundle on Z and let $\Omega^r_Z \to \mathscr{L}$ be a non-zero map for some $r > 0$. Then it uniquely extends to a (non-zero) map $\Omega^r_{Z_1} \to n^*\mathscr{L}$.

Apply Proposition B to $\Omega^r_L \to \mathscr{O}_X(K_{\mathscr{G}})_{|L}$.

More generally, Proposition B applies if $Z \subseteq X$ is invariant under $\mathscr{G}$.

Let $\mathscr{G} \cong \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus r}$ be foliation of degree zero on $\mathbb{P}^n$.
Then $\mathscr{G}$ is given by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$.
The singular locus of $\mathscr{G}$ is a linear subspace S of dimension $r-1$.
The closure of the leaf through a point $p \notin S$ is the r-dimensional linear subspace L of $\mathbb{P}^n$ containing both p and S.
And $K_L + S \sim_{\mathbb{Z}} K_{\mathscr{G}|L}$.

Observe that all « leaves » contain S!

Setup 3

X smooth projective variety over $\mathbb{C}$
$\mathscr{G}$ algebraically integrable foliation on X of rank $r > 0$

Then there is a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ {\scriptstyle\psi}\downarrow & & \\ Y & & \end{array}$$

where:
Z and Y projective and normal
β birational
ψ equidimensional, connected fibers
$y \in Y$ is a general point $\rightsquigarrow$ $\psi(Z_y)$ is the closure a leaf
the map $Z \to Y \times X$ induced by ψ and β is finite (and birational onto its image so that $Z \to (\psi \times \beta)(Z)$ is the normalization map)

$\rightsquigarrow$ « family of leaves » of $\mathscr{G}$.

The image in X of an irreducible component of a fiber of ψ is either contained in the singular locus of $\mathscr{G}$ or it is the closure of a leaf.

An adjunction formula for algebraically integrable foliations 1

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ {\scriptstyle\psi}\downarrow & & \\ Y & & \end{array}$$

$\mathscr{G}$ induces on Z the foliation $\mathscr{G}_Z$ given by ψ by construction.

$(\psi \times \beta)(Z) \subset Y \times X$ is invariant under the foliation on $Y \times X$ given by $p_X^*\mathscr{G} \subset p_X^* T_X \subset p_X^* T_X \oplus p_Y^* T_Y$

Proposition B then shows that there exists non-zero map

$$\Omega_Z^r \twoheadrightarrow \Omega_{Z/Y}^r \to \beta^*\mathscr{O}_X(K_{\mathscr{G}})$$

and hence a map

$$\mathscr{O}_Z(K_{\mathscr{G}_Z}) \cong \Omega_{Z/Y}^{[r]} \to \beta^*\mathscr{O}_X(K_{\mathscr{G}}).$$

An adjunction formula for algebraically integrable foliations 2

Fact (local computation):

$$\Omega^{[r]}_{Z/Y} \cong \mathscr{O}_Z(K_{Z/Y} - R(\psi))$$

with

$$R(\psi) := \sum_P (\psi^* P - (\psi^* P)_{\mathrm{red}})$$

where P runs through all prime divisors on Y.

Proposition C (Araujo - D. 2013)

There is a canonically defined effective (integral) Weil divisor Δ on Z such that

$$K_{\mathscr{G}_Z} + \Delta = K_{Z/Y} - R(\psi) + \Delta \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}.$$

Notice that Δ is β-exceptional.

The restriction of β to a general fiber F of ψ identifies with the normalization of the closure of a leaf and $\Delta_{|F}$ with Δ_F of Proposition A.

The boundary divisor Δ_F

Setup 4

X smooth projective variety over $\mathbb{C}$
$\mathscr{G}$ algebraically integrable foliation on X of rank $r > 0$
$n\colon F \to L \subseteq X$ normalization of the closure of a *general* leaf L of $\mathscr{G}$, (F, Δ_F) the corresponding log leaf

Then $\operatorname{Supp} \Delta_F \subseteq \operatorname{Exc} \beta \cap F$ ($\Delta_F = \Delta_{|F}$ and Δ is β-exceptional)

Equality holds!

Proposition D (Araujo - D. 2019)

If (F, Δ_F) is a general log leaf, then $\operatorname{Supp} \Delta_F = \operatorname{Exc} \beta \cap F$.

In particular, either $\mathscr{G}$ is regular along F or $\Delta_F \neq 0$ (L closure of a *general* leaf).

Corollary (Araujo - D. 2019)

If $\mathscr{G}$ is Fano, then $\Delta_F \neq 0$.

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ {\scriptstyle\psi}\downarrow & & \\ Y & & \end{array}$$

$\operatorname{Supp} \Delta_F \subseteq \operatorname{Exc} \beta \cap F$

E prime β-exceptional divisor on Z such that $\psi(E) = Y$. We have to show that $E \subseteq \operatorname{Supp} \Delta$.

$\beta(E)$ has dimension at least $r-1$ as $\beta_{|F}$ is finite and $\dim E \cap F = r-1$.

Taking hyperplane sections $\rightsquigarrow$ may assume that $r = 1$ since

$$(L, F, \Delta_F) \longleftrightarrow (L \cap H, F \cap \beta^{-1}(H), \Delta_{F|F\cap\beta^{-1}(H)}).$$

Recall

$$K_{\mathscr{G}_Z} + \Delta \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$$

Local on $X \rightsquigarrow$ may assume X affine and $K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$, $\mathscr{G} = \mathscr{O}_X \partial$.

$\rightsquigarrow \partial_Z$ is regular on $Z \setminus \operatorname{Supp} \Delta$.

We argue by contradiction and assume that $E \not\subseteq \operatorname{Supp} \Delta$.

$S := \operatorname{Sing} \mathscr{G}$ (zero set of ∂), invariant under ∂, namely $\partial(\mathscr{I}_S) \subseteq \mathscr{I}_S$.

f non-zero regular function on X, vanishing on S, such that $m := v_E(f)$ is minimal; $m > 0$ since $\beta(E) \subseteq S$.

g local equation of E on some open subset $U \subseteq Z \setminus \operatorname{Supp} \Delta$ (or $v_E(g) = 1$)

$\rightsquigarrow$ $f \circ \beta = ug^m$ with $v_E(u) = 0$

$\rightsquigarrow$ $\partial_Z(f \circ \beta) = g^m \partial_Z(u) + mug^{m-1}\partial_Z(g)$

Now, $v_E(\partial_Z(f \circ \beta)) = v_E(\partial(f)) \geqslant m$ ($\partial(f)$ vanishes on S) and $v_E(\partial_Z(u)) \geqslant 0$ (∂_Z is regular on $Z \setminus \operatorname{Supp} \Delta$)

If $v_E(\partial_Z(g)) = 0$, then $v_E(\partial_Z(f \circ \beta)) = m - 1$, a contradiction.

$\rightsquigarrow$ $v_E(\partial_Z(g)) \geqslant 1$. But then E is invariant under $\beta^{-1}\mathscr{G}$, a contradcition.

Miyaoka proved in 1993 that the anticanonical bundle of a smooth projective morphism $X \to C$ onto a smooth proper curve cannot be ample.

THEOREM (ARAUJO - D. - KOVÁCS 2008)

Let X be a normal projective variety, and $f : X \to C$ a surjective morphism with connected fibers onto a smooth curve. Let $\Delta \subseteq X$ be an effective Weil $\mathbb{Q}$-divisor. Suppose that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If (X, Δ) is log canonical over the generic point of C, then $-(K_{X/C} + \Delta)$ is not ample.

The statement is wrong if one drops the condition that (X, Δ) is log canonical over the generic point of C.

Assume to the contrary that (X, Δ) is log canonical over the generic point of C, and $-(K_{X/C} + \Delta)$ is ample.

Let $\pi : \tilde{X} \to X$ be a log resolution of singularities of (X, Δ), A an ample divisor on C, and $m \gg 0$ such that $D = -m(K_{X/C} + \Delta) - f^*A$ is very ample.

Write

$$K_{\tilde{X}} + \pi_*^{-1}\Delta = \pi^*(K_X + \Delta) + E_+ - E_-,$$

where E_+ and E_- are effective π-exceptional divisors with no common components and the support of $\pi_*^{-1}\Delta + E_+ + E_-$ is a snc divisor.

Set $\tilde{f} := f \circ \pi$ and let $\tilde{D} \in |\pi^* D|$ be a general member. Set also $\tilde{\Delta} = \pi_*^{-1}\Delta + \frac{1}{m}\tilde{D} + E_-$.

Then $(\tilde{X}, \tilde{\Delta})$ is log canonical over the generic point of C and that

$$K_{\tilde{X}} + \tilde{\Delta} \sim_{\mathbb{Q}} \tilde{f}^* K_C + E_+ - \frac{1}{m}\tilde{f}^* A.$$

Since E_+ is effective and π-exceptional, $\pi_* \mathscr{O}_{\tilde{X}}(lE_+) = \mathscr{O}_X$ for any $l \in \mathbb{N}$ such that lE_+ is integral.

Then for any sufficiently divisible $l \in \mathbb{N}_{>0}$,

$$\tilde{f}_* \mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C} + \tilde{\Delta})) \cong \tilde{f}_* \mathscr{O}_{\tilde{X}}(l(mE_+ - \tilde{f}^* A)) \cong \mathscr{O}_C(-lA).$$

But $\tilde{f}_* \mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C} + \tilde{\Delta}))$ is nef (Fujita 1977, ... , Campana 2004), but that contradicts the fact that A is ample.

Let $\mathscr{G} \cong \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus r}$ be foliation of degree zero on $\mathbb{P}^n$. Then $\mathscr{G}$ is given by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$.
The singular locus of $\mathscr{G}$ is a linear subspace S of dimension $r-1$.
The closure of the leaf through a point $p \notin S$ is the r-dimensional linear subspace L of $\mathbb{P}^n$ containing both p and S.
And $K_L + S \sim_{\mathbb{Z}} K_{\mathscr{G}|L}$.

Any « leaf » contains $\operatorname{Supp} \Delta_L = S$.

Proposition E (Araujo - D. 2016)

Let $\mathscr{G}$ be an algebraically integrable Fano foliation on a complex projective manifold X. Suppose that a general log leaf (F, Δ_F) has log canonical singularities.

Then there is a closed irreducible subset $T \subseteq X$ satisfying the following property. For a general log leaf (F, Δ_F) of $\mathscr{G}$, there exists a log canonical center S of (F, Δ_F) whose image in X is T.

In particular, there is a common point in the closure of a general leaf of $\mathscr{G}$.

Idea of proof 1

We « show » that there is a common point in the closure of a general leaf of $\mathscr{G}$.

Let

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ {\scriptstyle\psi}\big\downarrow & & \\ Y & & \end{array}$$

be the family of leaves.

We argue by contradiction.

If $C \subseteq Y$ is a general complete intersection curve, then the restriction of β to $S := \beta^{-1}(C)$ is a finite morphism.

There exists a finite morphism $C_1 \to C$ with C_1 smooth and projective such that $S_1 \to C_1$ has reduced fibers, where S_1 is the normalization of $S \times_{C_1} C$.

Idea of proof 2

Proposition B then implies that there exists an effective (integral) Weil divisor Δ_{S_1} on S_1 such that

$$K_{S_1/C_1} + \Delta_{S_1} \sim_{\mathbb{Z}} K_{\mathscr{G}|S_1}.$$

By assumption, (S_1, Δ_{S_1}) is log canonical over the generic point of C_1.

On the other hand, $-(K_{S_1} + \Delta_{S_1})$ is ample since $S_1 \to X$ is finite by construction.

But this contradicts the theorem.

Thanks!