Holomorphic Poisson structures

Discussion session

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Problem 1: projective embedding

Definition

A **Poisson projective embedding** of a complex Poisson manifold (X, π_X) is the following data:

- a positive integer N
- a Poisson bivector $\pi_{\mathbb{P}^N}$ on $\mathbb{P}^N$
- a closed embedding $X \hookrightarrow \mathbb{P}^N$ such that $\pi_{\mathbb{P}^N}|_X = \pi_X$

Problem

When does (X, π) admit a Poisson projective embedding?

Example: If $X \subset \mathbb{P}^3$ is a smooth surface of degree d, then a nonzero Poisson structure on X can be extended to $\mathbb{P}^3$ if and only if $d \neq 4$. Case $d = 4$ and $N > 3$?

Possible approach for Fanos: linear systems from Poisson line bundles $\mathcal{L} \to X$, especially $\mathcal{L} = K_X$; see [4, Sections 5 and 12] and [5, Section 7.4]

Problem 2: log symplectic singularities

$\dim X = 2n$ $\qquad$ $0 \neq \pi^n \in H^0(X, K_X^{-1})$ $\qquad$ $Y = Zeros(\pi^n)$ reduced

Problem

What kind of singularities can Y have?

Some known facts/examples:

- Theorem ([2, 6]): if Y is singular, then $\operatorname{codim}(Y_{sing}, X) \leq 3$
- Theorem (Weinstein splitting): near a codim two symplectic leaf

$$Y \sim (\text{any plane curve germ}) \times (\text{smooth})$$

- Theorem ([6]): if Y_{sing} is a codimension-three modular leaf, then

$$Y \sim (\text{simple elliptic surface } \tilde{E}_n) \times (\text{smooth}) \qquad n = 6, 7, 8$$

e.g. if $X = \mathbb{P}^4$ then have Feigin–Odesskii structure $\pi = q_{5,1}$ (case $\tilde{E}_6$)

Problem

Construct/classify smooth log symplectic Fano fourfolds (X, π) with Y of type $\tilde{E}_n$ with $n = 6, 7, 8$. What if we allow X to have "mild" singularities?

Problem 3: moduli spaces and stability

X smooth projective

$$Pois(X) := \left\{\pi \in H^0(\wedge^2 \mathcal{T}_X) \,\middle|\, [\pi,\pi] = 0\right\} \circlearrowleft Aut(X)$$

$$\text{substack } [Pois(X)/Aut(X)] \hookrightarrow \mathcal{M}_{\mathrm{Pois}}$$

Problem

Construct a "nice" moduli space, e.g. by introducting a suitable GIT/K-stability condition on pairs (X,π).

e.g. for $X = \mathbb{P}^2$, moduli of GIT-semistable cubic curves $\cong \mathbb{P}^1$ via j-invariant

Problem

Poisson Fano threefolds with $b_2(X) = 1$ were classified in [1, 3]. Which elements in $Pois(X)$ are GIT (semi-)stable?

References

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