Holomorphic Poisson structures

Part IV of IV: Global aspects, non-regular case

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Which manifolds admit nontrivial Poisson structures?

Corollary (End of last lecture)

If $c_1(X)^{c+1} \neq 0$, then $\nexists$ a regular Poisson structure of corank c on X.

NB: $c_1(X) \in H^2(X)$ measures curvature

Notation: $c_1(X) > 0 \iff \int_Y c_1(X)^{\dim(Y)} > 0$ $\forall$ closed subvarieties $Y \subset X$

Fundamental building blocks of algebraic geometry:

- $c_1(X) > 0$: Fano, e.g. $\mathbb{P}^n$, Grassmannians
- $c_1(X) = 0$: Calabi–Yau, e.g. K3, tori, IHSMs
- $c_1(X) < 0$: general type, e.g. curves of genus > 1

Poisson building blocks

General type: $c_1(X) < 0$

$$H^0(\wedge^2 \mathcal{T}_X) = H^0(\Omega_X^{n-2} \otimes K_X^{-1}) = 0 \qquad \text{(Kodaira–Nakano)}$$

Only Poisson structure is $\pi = 0$.

Calabi–Yau: $c_1(X) = 0$

$$\tilde{X} \cong \mathbb{C}^n/\Lambda \times \text{IHSMs} \times (Y, \pi_Y = 0) \qquad \text{(Beauville–Bogomolov)}$$

All Poisson structures are regular.

Fano: $c_1(X) > 0$

???

$c_1(X)^{c+1} \neq 0$ for all $c < \dim(X)$, so π regular $\implies \pi = 0$

Degeneracy loci

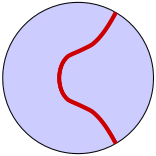
$$X_0 \subset X_2 \subset X_4 \subset \cdots \subset X$$

$$\begin{aligned} X_{2k} &:= \text{union of symplectic leaves of dimension } \leq 2k \\ &= \{x \in X \mid \operatorname{rank}(\pi|_x) \leq 2k\} \\ &= Zeros(\pi^{k+1}) \end{aligned}$$

Example (Surfaces)

X surface $\qquad Y = Zeros(\pi) \qquad Y = X_0 \subset X_2$

$\dim X_0 = 1 > 0 \qquad [Y] = c_1(X) \in H^2(X)$



Example (Generically symplectic)

π generically nondegenerate $\iff 0 \neq \pi^n \in H^0(K_X^{-1})$

$\iff X = X_{2n} \supsetneq X_{2n-2} =: Y$

NB: $Y \subset X$ anticanonical, $\dim X_{2n-2} > 2n-2$ if nonempty, $[Y] = c_1(X)$.

Maximal degeneracy locus

Suppose that $\max \operatorname{rank}(\pi) = 2r$, i.e. $X = X_{2r}$

$$\text{maximal degeneracy locus: } X_m \qquad m := 2r - 2$$

$$U := X \setminus X_m \qquad \text{open, regular, corank } c := \dim X - 2r$$

Proposition

If $X = X_{2r}$ and $c_1(X)^{c+1} \neq 0$, then $\dim X_m > m$

Proof.

Let $c' = \operatorname{codim}(X_m)$. Need to show $c' \leq c + 1$.
Consider restriction:

$$\begin{array}{rcll} H^{2j}(X) & \to & H^{2j}(U) & \text{injective for } j < c' \\ c_1(X)^j & \mapsto & c_1(U)^j & = 0 \text{ for } j > c \end{array}$$

Thus $c_1(X)^j = 0$ for $c < j < c'$, but nonzero for $j = c + 1$.
Only possible if $c' \leq c + 1$. □

Bondal's conjecture

X Fano, $\dim = n \implies c_1(X)^j \neq 0$ for $j \leq n$

$\dim X$	$\dim X_0$	
n	naive: $n - \binom{n}{2}$	Bondal
0	0	0
1	1	1
2	1	1
3	0	1
4	-2	1
⋮	⋮	⋮

Conjecture (Bondal 1993)

If X Fano and $2r < \dim X$, then X_{2r} has an irreducible component of dimension $> 2r$.

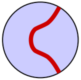
Conjecture has been proven for:

- Points and curves: trivial
- Surfaces: $[X_0] = c_1(X)$
- Threefolds: previous proposition
- Fourfolds: Gualtieri–P. 2013, uses modular vector field $\Delta\pi$
  - Case $X = X_2$: previous proposition
  - Case $X = X_4$, and $Y = X_2$ anticanonical divisor (threefold)
    - ⋆ Problem: transversality of $\Delta\pi$ obstructs Bott vanishing in Y
    - ⋆ Show Y singular in codimension two, $[Y_{sing}] = c_1c_2 - c_3$
    - ⋆ Show obstruction vanishes on Y_{sing}, run Bott vanishing there

Fano surfaces = "del Pezzo surfaces"

degree $d := \int_X c_1(X)^2$ $\qquad b_2(X) = 10 - d$ $\qquad 1 \leq d \leq 9$

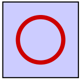
$d = 9$ $\qquad d = 8$ $\qquad d = 7$ $\qquad \cdots$ $\qquad d = 1$



$\mathbb{P}^2 \longleftarrow \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$

$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \leftarrow dP(7) \longleftarrow \cdots \longleftarrow dP(1)$

$dP(7) \to \mathbb{P}^1 \times \mathbb{P}^1$



$\mathbb{P}^1 \times \mathbb{P}^1$

Arrows are blowups $\implies$ reduce classification of (X, π) to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$:

Lemma

Suppose surface X' is the blowup of X at $p \in X$. Then

$$\{\pi' \text{ on } X'\} \cong \{\pi \text{ on } X \text{ such that } \pi|_p = 0\}$$

Families of del Pezzos

Suppose $f : X \dashrightarrow Z$ with del Pezzos as fibres. Set $K_{X/Z}^{-1} = K_X^{-1} \otimes f^* K_Z$ anticanonical of fibres.

$$0 \neq \pi \in H^0(K_{X/Z}^{-1}) \qquad \text{Poisson, generic rank two, leaves} \subset \text{fibres}$$

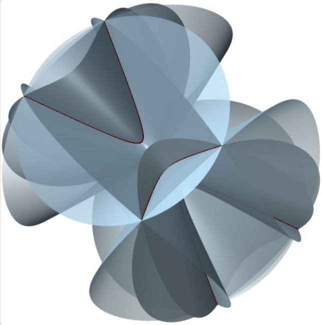
Example (Sklyanin 1982, pencil of quadrics)

$$X = \mathbb{P}^3 \qquad Z = \mathbb{P}^1 \qquad f = \underbrace{[Q_0 : Q_1]}_{\text{quadrics}}$$

$$f^{-1}([s:t]) = \{tQ_0 - sQ_1 = 0\} \cong \mathbb{P}^1 \times \mathbb{P}^1$$

$$K_{X/Z}^{-1} = \mathcal{O}(4) \otimes f^*\mathcal{O}(-2) \cong \mathcal{O}$$

$$X_0 = \underbrace{\{Q_0 = Q_1 = 0\}}_{\substack{\text{elliptic curve} \\ \Delta_\mu \pi \neq 0}} \cup \underbrace{\{p_1, p_2, p_3, p_3\}}_{\cong \text{ origin in } \mathfrak{sl}_2}$$



Further constructions

Many ways to construct Poisson varieties (often Fano):

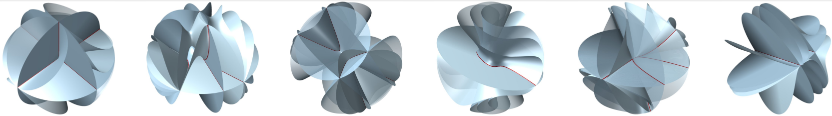
- Blowing up/down (Polishchuk 1997)
- Forms with poles on anticanonical $Y \subset X$
  - "Log" symplectic forms (Goto 2003) $\rightsquigarrow$ open symplectic leaf $X \setminus Y$
  - closed $(n-2)$-form $\rightsquigarrow$ generic rank two
- Projective bundles
  - $\mathbb{P}^1$-bundles $X \to (Y, \pi)$ with "flat Poisson connection" (Polishchuk 1997)
  - $\mathbb{P}^n$-bundles $X \to (Y, \pi = 0)$ via co-Higgs fields (Matviichuk 2020)
- Lie-theoretic constructions (e.g. works of J. H. Lu)
  - Flag manifolds
  - Bott–Samelson varieties
  - $G \circlearrowright X$, use "r-matrix" $r \in \wedge^2\mathfrak{g} \to H^0(\wedge^2 \mathcal{T}_X)$
  - ...
- Moduli spaces: bundles on elliptic curves (Feigin–Odesskii, ...), monopoles (Atiyah–Hitchin, ...), Higgs bundles (Hitchin, ...), sheaves on Poisson surfaces (Bottacin, ...)
- ...

Higher-dimensional classification?

$$\mathcal{M}_{\mathsf{Pois}} := \frac{\{(X,\pi)\}}{\sim} = \cup_i \mathcal{M}_{\mathsf{Pois},i} \qquad \mathcal{M}_{\mathsf{Pois}}(\mathbb{P}^2) \cong \frac{H^0(\mathbb{P}^2, \mathcal{O}(3))}{PGL_3}$$

Theorem (Cerveau–Lins Neto 1996, Loray–Pereira–Touzet 2013)

$\mathcal{M}_{\mathsf{Pois}}(\mathbb{P}^3)$ *has 6 irreducible components. Sim. results for Fanos w/* $b_2 = 1$.



$b_2 > 1$? *dim* > 3? — some examples of components of $\mathcal{M}_{\mathsf{Pois}}$ are known (Lima–Pereira, Okumura, P., P. Schedler, Ran)

Theorem (Matviichuk–P.–Schedler in prep.)

$\mathcal{M}_{\mathsf{Pois}}(\mathbb{P}^4)$ *has approx.* 40 *(exact number TBC) components corresponding to Poisson structures admitting generically symplectic toric degenerations.*