Holomorphic Poisson structures

Part III of IV: Global aspects, regular case

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Regular Posson manifolds

Foliations of (X, π):

$$\underbrace{\mathcal{F}_\pi}_{\text{symplectic foliation}} \subset \underbrace{\mathcal{F}_\pi^{mod}}_{\text{modular foliation}} \subset \mathcal{T}_X$$

symplectic foliation (span of Hamiltonians)

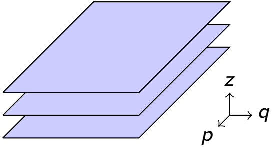
modular foliation ($\mathcal{F}_\pi$ + modular $\zeta = \Delta_\mu \pi$)

Definition

A Poisson manifold (X, π) is **regular** if rank(π) is constant.

Local picture: Darboux–Weinstein p_i, q^i, z^j

$$\pi = \sum_i \partial_{q^i} \wedge \partial_{p_i} \qquad \zeta = \Delta_\mu \pi = 0$$



Symplectic leaves = modular leaves

Global picture for X compact: are leaves compact? classification?

Constructions

Example

X any complex manifold $\implies$ $\pi = 0$ regular

Example

X holomorphic symplectic $\implies$ π nondegenerate, hence regular

Example

(X, π_X), (Y, π_Y) regular $\implies$ $(X \times Y, \pi_X + \pi_Y)$ regular

Example

X regular $\implies$ any covering space $\tilde{X}$ regular

Example

G discrete group, $G \circlearrowright X$ freely by Poisson isomorphisms
such that X/G manifold $\implies$ X/G regular

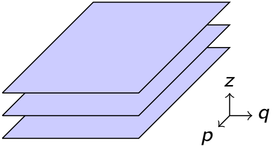
Abelian varieties

$V \cong \mathbb{C}^n$ vector space

$\pi \in \wedge^2 V$ constant bivector

$W := \mathrm{img}(\pi^\sharp) \subset V$

symplectic leaves: $L_v := W + v, v \in V$



$\mathbb{Z}^{2n} \cong \Lambda \hookrightarrow V$
full rank lattice

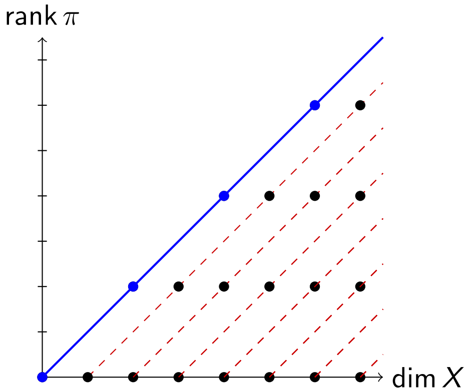
$X := V/\Lambda \cong (S^1)^{2n}$
compact complex torus
(abelian variety if projective)

$$\text{leaf through } [v] \in X \quad = \quad \mathrm{img}(L_v) \cong \frac{W}{W \cap \Lambda} + [v]$$

leaves of X are compact $\iff$ $\Lambda \cap W \subset W$ full rank

NB: if $W \neq V$, then generically $\Lambda \cap W = 0$, so leaves $\cong W$!

Rank vs. dimension



Useful to organize classification by

$$\dim X \qquad \text{and} \qquad \operatorname{corank}(\pi) := \dim(X) - \operatorname{rank}(\pi)$$

$$\operatorname{corank}(\pi) = 0 \iff \text{holomorphic symplectic}$$

Compact symplectic surfaces

X compact connected symplectic surface

$\pi \in K_X^{-1}$ nonvanishing $\qquad K_X \cong \mathcal{O}_X$

NB: $H^0(\mathcal{O}_X) = \mathbb{C}$, so π is unique up to rescaling

Theorem (Part of Enriques–Kodaira classification of surfaces)

X is isomorphic to one of the following:

- *complex torus* $\mathbb{C}^2/\Lambda$
- *primary Kodaira surface: $X \to Y$ with elliptic curves as base and fibre*
- *K3 surface: simply connected, diffeo. to* $\{\text{quartic}\} \subset \mathbb{P}^3$

Compact holomorphic symplectic manifolds

Example

(Y_i, ω_i) compact symplectic surfaces, $X = Y_1 \times \cdots \times Y_n$

$$\omega_X = \lambda_1 p_1^* \omega_1 + \cdots + \lambda_n p_n^* \omega_n \qquad \lambda_i \in \mathbb{C}^*$$

symplectic form on X depending nontrivially on n parameters

Definition

An **irreducible holomorphic symplectic manifold (IHSM)** is a compact Kähler manifold X with a holomorphic symplectic structure ω that is unique up to rescaling:

$$H^0(\Omega_X^2) = \mathbb{C} \cdot \omega$$

Theorem (Part of Beauville–Bogomolov decomposition)

Every compact Kähler holomorphic symplectic manifold has an étale cover that is a product of IHSMs and symplectic tori $\mathbb{C}^{2r}/\Lambda$.

Symplectic surfaces $\rightsquigarrow$ IHSMs

Example (Beauville)

Consider a K3 surface Y, construct

$$\underbrace{Sym^n(Y) := Y^n/S_n}_{\substack{\text{singular Poisson variety} \\ \text{smooth locus symplectic}}} \longleftarrow \underbrace{Hilb^n(Y)}_{\substack{\text{Hilbert scheme, smooth} \\ \text{IHSM}}}$$

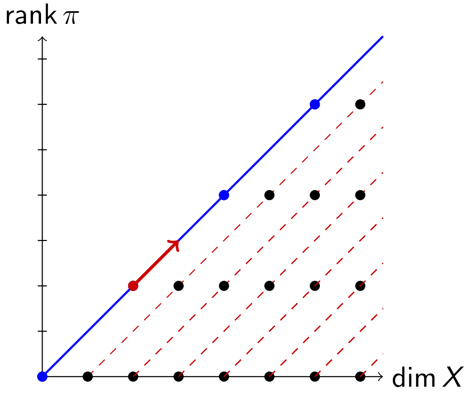
Example (Beauville)

Y an abelian surface, fibre of $+ : Hilb^n(Y) \to Y$ is an IHSM

Example (Mukai, O'Grady)

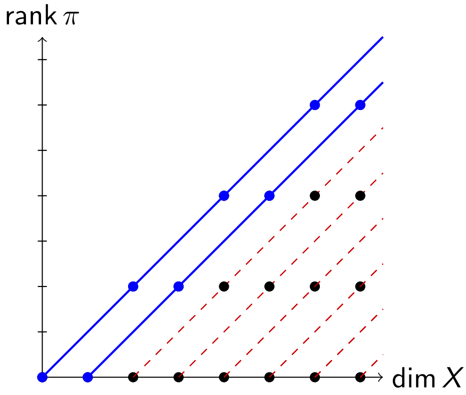
Mukai: moduli of sheaves on a K3/abelian surface is (singular) symplectic.
O'Grady: desingularization gives IHSMs of dimension 6, and 10.

Last I heard, all other known IHSMs are deformations of those examples.



Open question: can we find others?

Corank one



$\text{corank}(\pi) = 1$

$\dim X = \text{rank}(\pi) + 1 \quad$ odd

Corank one: examples

Example (Products with curves)

(Y, π) symplectic, Z a curve $\implies$ $X = Y \times Z$ regular corank one

leaves: $Y \times \{z\}$ for $z \in Z$

Example (Products with tori)

$X = Y \times T$ with Y symplectic and $T = \mathbb{C}^{2r+1}/\Lambda$ corank one

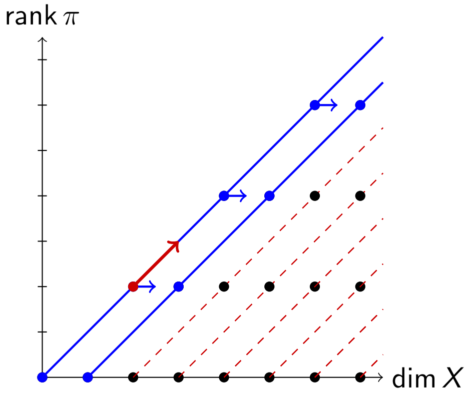
Example (Flat $\mathbb{P}^1$-bundles)

Y symplectic, $X \to Y$ a flat $\mathbb{P}^1$-bundle, π_X the horizontal lift of π_Y.

Theorem (Druel 1999 for dim $X = 3$; Touzet 2008, Loray–Pereira–Touzet 2018 for dim X arbitrary)

X compact Kähler, corank$(\pi) = 1$ *away from a subset of codim* ≥ 3 $\implies$ *π regular and (X, π) is étale-covered by one of these examples*

The landscape



Arbitrary corank: a topological constraint

Recall: every vector bundle $\mathcal{E} \to X$ has a Chern character

$$\mathrm{ch}(\mathcal{E}) = \sum_{j \geq 0} \mathrm{ch}_j(\mathcal{E}) \qquad \mathrm{ch}_j(\mathcal{E}) \in H^{2j}(X;\mathbb{Q})$$

Useful properties:

- depends only on isomorphism class of $\mathcal{E}$
- $\mathrm{ch}(\mathcal{E}_1 \oplus \mathcal{E}_2) = \mathrm{ch}(\mathcal{E}_1) + \mathrm{ch}(\mathcal{E}_2)$
- $\mathrm{ch}_j(\mathcal{E}^\vee) = (-1)^j \mathrm{ch}_j(\mathcal{E})$
- $\mathrm{ch}(f^*\mathcal{E}) = f^*\mathrm{ch}(\mathcal{E})$
- $\mathrm{ch}_j(\mathcal{E}) = 0$ for $j > \dim_{\mathbb{C}} X$, since $H^{2j} = 0$

For (X, π) Poisson, consider $\mathrm{ch}(X) := \mathrm{ch}(\mathcal{T}_X)$

Example

X symplectic $\implies \mathcal{T}_X \cong \mathcal{T}_X^\vee \implies \mathrm{ch}_j(X) = 0$ for j odd

Example

$f : (X, \pi) \to Z$, fibres = symplectic leaves, $c :=$ corank$(\pi) = \dim(Z)$

$$0 \longrightarrow \mathcal{F}_\pi \longrightarrow \mathcal{T}_X \longrightarrow f^*\mathcal{T}_Z \longrightarrow 0$$

$$\mathrm{ch}_j(X) = \underbrace{\mathrm{ch}_j(\mathcal{F}_\pi)}_{=0 \text{ if } j \text{ odd}} + \underbrace{f^*\mathrm{ch}_j(Z)}_{=0 \text{ if } j>c}$$

Theorem (Example above + Bott vanishing)

Suppose (X, π) *is regular with* corank$(\pi) = c$,

$$P = P(\mathrm{ch}_1, \mathrm{ch}_3, \ldots) \in H^{2k}(X; \mathbb{Q})$$

is a polynomial of total degree $k > c$, *then* $P = 0$.

Corollary

If $c_1(X)^{c+1} \neq 0$, *then* $\nexists$ *a regular Poisson structure of corank* c *on* X.