# Holomorphic Poisson structures

## Part II of IV: Foliations of Poisson manifolds

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# Last time



$$\text{Poisson bracket } \{-,-\} \text{ on } \mathcal{O}_X \iff \begin{cases} \pi \in H^0(\wedge^2 \mathcal{T}_X) \\ 0 = [\pi, \pi] \in H^0(\wedge^3 \mathcal{T}_X) \end{cases}$$

$$\text{coordinates } x^i \qquad \pi = \sum_{i<j} \pi^{ij} \partial_{x^i} \wedge \partial_{x^j} \qquad \pi^{ij} = \{x^i, x^j\} \in \mathcal{O}_X$$

$$\{f, g\} = \sum_{i<j} \pi^{ij} \cdot \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial x^j} \right)$$

**Theorem (Weinstein splitting)**

*Every point $x \in (X, \pi)$ has a neighbourhood isomorphic to the product of a nondegenerate (i.e. symplectic) structure and a Poisson structure that vanishes at a point:*

$$\pi = \sum_{i=1}^{r} \partial_{q^i} \wedge \partial_{p_i} + \tilde{\pi}(z) \qquad \tilde{\pi}|_x = 0$$

# A foliation



Recall: anchor map $\pi^\sharp : \Omega^1_X \to \mathcal{T}_X$ sending $\alpha \mapsto \iota_\alpha \pi$.

Every $f \in \mathcal{O}_X$ has Hamiltonian vector field $\xi_f = \{f, -\} = \pi^\sharp(df)$

Equivalence relation on $X$:

$$x \sim x' \qquad \iff \qquad x, x' \text{ joined by a sequence of Hamiltonian flows}$$

Equivalence classes:

$$X = \bigsqcup_j L_j$$

Claim: $L_j$ are immersed analytic submanifolds, giving possibly singular foliation of $X$

$$\begin{aligned} \mathcal{F}_\pi &:= \mathcal{O}_X \cdot \{\xi_f \mid f \in \mathcal{O}_X\} \subset \mathcal{T}_X \\ &= \mathrm{img}(\pi^\sharp : \Omega^1_X \to \mathcal{T}_X) \end{aligned}$$

$$\text{Jacobi identity} \iff [\xi_f, \xi_g] = \xi_{\{f,g\}} \implies [\mathcal{F}_\pi, \mathcal{F}_\pi] \subset \mathcal{F}_\pi$$

# Symplectic leaves



$$L \subset X \text{ leaf of } \mathcal{F}_\pi \quad \Longrightarrow \quad \mathcal{T}_L = \mathrm{img}(\pi^\sharp|_L) \subset \mathcal{T}_X|_L$$

$$\Longrightarrow \quad \pi|_L \in \wedge^2\mathcal{T}_L \subset \wedge^2\mathcal{T}_X|_L \text{ nondegenerate}$$

$$\Longrightarrow \quad \omega_L := \pi|_L^{-1} \in H^0(\Omega^2_L) \text{ symplectic}$$

**Theorem**

*If $(X, \pi)$ is any Poisson variety, then every leaf of $\mathcal{F}_\pi$ carries a canonical holomorphic symplectic form.*

**Definition**

$\mathcal{F}_\pi$ is called the **symplectic foliation** of $(X, \pi)$.

# Examples



## Example (Trivial)

$\pi = 0 \implies \mathcal{F}_\pi = 0 \implies$ every $p \in X$ is a symplectic leaf with $\omega_p = 0$

## Example (Nondegenerate)

$\pi$ nondegenerate $\implies \mathcal{F}_\pi = \mathcal{T}_X \implies$ leaves are the connected components of $X$, symplectic form $\omega = \pi^{-1}$

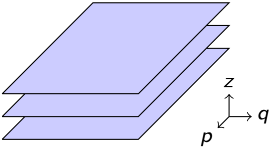
## Example (Constant)

$$X = \mathbb{C}^n_{p_i, q^i, z^j} \qquad \pi = \sum \partial_{q^i} \wedge \partial_{p_i}$$

$$\mathcal{F}_\pi = \mathrm{span}\{\partial_{p_i}, \partial_{q_i}\}$$

Symplectic leaves:

$$L = \mathbb{C}^{2r} \times \{z = const\} \qquad \omega_L = \sum_i dp_i \wedge dq^i|_L$$



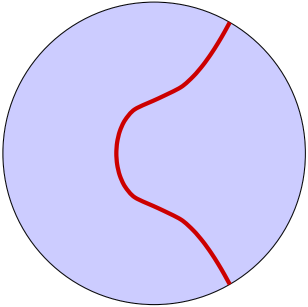
# Example: surfaces

$(X, \pi)$ smooth connected surface:

$$\pi \in H^0(\wedge^2 \mathcal{T}_X) = H^0(K_X^{-1})$$

Anticanonical divisor:

$$Y = Zeros(\pi) \subset X$$



Symplectic leaves:

- dimension two: $X \setminus Y$ with $\omega = \pi|_{X \setminus Y}^{-1}$
- dimension zero: points of $Y$ with $\omega = 0$

# Example: Lie algebras



$$\mathfrak{g} = Lie(G) \qquad X = \mathfrak{g}^\vee \qquad f \in \mathfrak{g} \subset \mathcal{O}(X)$$

$$\{f, -\} : \mathcal{O}(X) \to \mathcal{O}(X) \qquad \text{extends} \qquad ad_f = [f, -] : \mathfrak{g} \to \mathfrak{g}$$

**Theorem (Kirillov–Kostant–Soriau)**

*Symplectic leaves of $\mathfrak{g}^\vee$ are the orbits of the coadjoint action $G \circlearrowright \mathfrak{g}^\vee$. In particular, each coadjoint orbit carries a canonical symplectic form.*

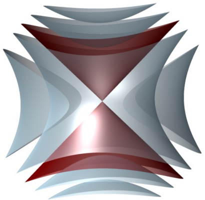
**NB:** $\{0\} \subset \mathfrak{g}^\vee$ always a leaf

**Example ($\mathfrak{g} = \mathfrak{sl}(2; \mathbb{C})$)**

$$\mathfrak{g}^\vee \cong \mathbb{C}^3_{e,f,h} \qquad \mathfrak{g} \cong \mathfrak{g}^\vee \text{ via } A \mapsto Tr(A-)$$

$$0 \neq A, B \text{ in same leaf} \iff \det(A) = \det(B)$$

$$\text{leaves:} \qquad \{0\}, \{h^2 + 4ef = \text{constant}\}$$



$\pi$ is invariant under Hamiltonian flows: $\mathscr{L}_{\xi_f}\pi = 0$, indeed

$$\mathscr{L}_{\xi_f}\pi = [\xi_f, \pi] = [\iota_{df}\pi, \pi] = -[[f, \pi], \pi] = -[f, \underbrace{[\pi, \pi]}_{=0}] - [\pi, \iota_{df}\pi] = -\mathscr{L}_{\xi_f}\pi$$

### Definition

A vector field $\xi \in \mathcal{T}_X$ is a **Poisson vector field** if $\mathscr{L}_\xi \pi = 0$.

**NB:** $\xi \in \mathcal{T}_X$ locally Hamiltonian $\implies$ $\xi$ Poisson

### Lemma

*If $\pi$ nondegenerate, then $\xi \in \mathcal{T}_X$ Poisson $\iff$ $\xi$ is locally Hamiltonian.*

### Proof.

Given $\xi \in \mathcal{T}_X$. Nondegeneracy $\implies$ $\xi = \pi^\sharp(\alpha)$ where $\alpha \in \Omega^1_X$.
Computation $\implies$ $\mathscr{L}_\xi \pi = 0$ if and only if $d\alpha = 0$. Poincaré lemma $\implies$ locally $\alpha = df$, so $\xi = \xi_f$. □

# Non-Hamiltonian symmetries



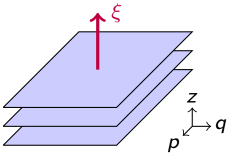
**Example (locally Hamiltonian $\not\Rightarrow$ globally Hamiltonian)**

$\pi$ nondegenerate, $\alpha \in H^0(\Omega^1_X)$ closed but not exact $\implies \xi = \pi^\sharp(\alpha)$ Poisson but not globally Hamiltonian.

**Example (Poisson $\not\Rightarrow$ locally Hamiltonian)**

$$X = \mathbb{C}^3 \qquad \text{coords } p, q, z \qquad \pi = \partial_q \wedge \partial_p$$

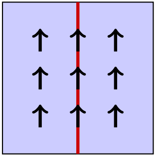
Poisson vector field $\xi = \partial_z$ transverse to leaves




**Example (locally Hamiltonian near a point $\not\Rightarrow$ locally Hamiltonian everywhere)**

$$X = \mathbb{C}^2_{u,v} \qquad \pi = u\partial_u \wedge \partial_v \qquad \xi = \partial_v$$

$\partial_v = \xi_{\log u}$ for $u \neq 0$ but not $u = 0$.



# Modular vector field



For $X$ smooth, $\dim X = n$, have symmetries generated by volume forms:

$$0 \neq \mu \in \Omega^n_X = K_X$$

$$\Omega^\bullet_X \xleftrightarrow{\sim} \wedge^{n-\bullet}\mathcal{T}_X$$

$d$ (degree $+1$) acts on $\Omega^\bullet_X$; divergence $\Delta_\mu$ (degree $-1$) acts on $\wedge^{n-\bullet}\mathcal{T}_X$.

### Definition (Brylinski–Zuckerman, Polishchuk, Weinstein 1997)

If $(X,\pi)$ Poisson manifold, $\mu \in K_X$, the vector field

$$\zeta := \Delta_\mu \pi \in \mathcal{T}_X$$

is called the **modular vector field** of $\pi$ with respect to $\mu$.

**Meaning:** $\Delta_\mu \pi = 0 \iff \mu$ invariant under all Hamiltonian flows

# Properties of the modular vector field 

$$\zeta := \Delta_\mu \pi$$

Local expression:

$$\pi = \sum \pi^{ij} \partial_{x^i} \wedge \partial_{x^j} \qquad \mu = dx^1 \wedge \cdots \wedge dx^n \qquad \zeta = \sum_{ij} \frac{\partial \pi^{ij}}{\partial x^i} \partial_{x_j}$$

It is a Poisson vector field:

$$\mathscr{L}_\zeta(\pi) = 0$$

Change of volume form:

$$\begin{aligned} \mu \rightsquigarrow \mu' &= g\mu \\ \zeta \rightsquigarrow \zeta &= \zeta' - \xi_{\log g} \end{aligned}$$

So $\zeta$ is "locally well-defined modulo Hamiltonians"

# Modular foliation



Introduce

$$\mathcal{F}_\pi \subset \mathcal{F}_\pi^{mod} \subset \mathcal{T}_X$$

where locally

$$\mathcal{F}_\pi^{mod} = \mathcal{F}_\pi + \mathcal{O}_X \cdot \Delta_\mu \pi$$

Check:

$$[\mathcal{F}_\pi^{mod}, \mathcal{F}_\pi^{mod}] \subset \mathcal{F}_\pi^{mod}$$

**Definition**

The foliation induced by $\mathcal{F}_\pi^{mod}$ is called the **modular foliation of** $(X, \pi)$.

Locally: leaves $L$ of $\mathcal{F}_\pi^{mod}$ = orbits of symplectic leaves under $\zeta$

- $\dim L$ even: symplectic leaf to which $\zeta$ is tangent
- $\dim L$ odd: one-parameter family of symplectic leaves related by $\zeta$

# Examples



**Example (Nondegenerate)**

$\pi$ nondegenerate, $\omega = \pi^{-1} \implies \mathcal{F}_\pi = \mathcal{F}_\pi^{mod} = \mathcal{T}_X$.

NB: $\mu = \omega^n \in H^0(K_X)$ Hamiltonian-invariant, so $\zeta = \Delta_\mu \pi = 0$ globally

**Example (Constant rank)**

rank($\pi$) constant $\implies$ locally $\pi = \sum_i \partial_{q^i} \wedge \partial_{p_i} \implies \zeta = 0$ in these coords $\implies \mathcal{F}_\pi^{mod} = \mathcal{F}_\pi$

**Example ($\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$)**

$$X = \mathfrak{g}^\vee = \mathbb{C}^3_{e,f,h} \qquad \pi = h\partial_e \wedge \partial_f + 2e\partial_h \wedge \partial_e - 2f\partial_h \wedge \partial_f$$

$$\mu = de \wedge df \wedge dh \qquad \Delta_\pi \mu = 0$$

$$\mathcal{F}_\pi^{mod} = \mathcal{F}_\pi \qquad \text{coadjoint orbits}$$

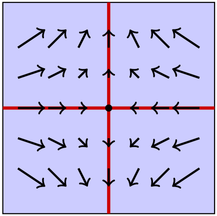
# Modular foliation of surfaces

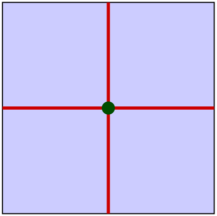


X smooth connected Poisson surface, $Y = Zeros(\pi) \subset X$ anticanonical

$$\pi = f(u,v)\partial_u \wedge \partial_v \qquad \zeta = f_u \partial_v - f_v \partial_u$$

e.g. if $f = uv$:





In general, modular leaves $L \subset X$:

- $\dim L = 2$: symplectic leaf $L = X \setminus Y$
- $\dim L = 1$: connected components of smooth locus $Y \setminus Y_{sing}$
- $\dim L = 0$: singular points of $Y$