Holomorphic Poisson structures

Part I of IV: Basic Poisson geometry

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Origin: Hamiltonian mechanics

Particle in 1d:

position	momentum	energy
q	$p = m \cdot \dot{q}$	$H(q,p) = \frac{p^2}{2m} + V(q)$

Hamilton's equations of motion:

$$\dot{q} = \frac{p}{m} = \frac{\partial H}{\partial p} \qquad \dot{p} = \text{force} = -\frac{\partial V}{\partial q} = -\frac{\partial H}{\partial q}$$

In general, $f(q,p)$ evolves according to

$$\dot{f} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} =: \{f, H\}$$

using Poisson bracket

$$\{-,-\} = \partial_q \wedge \partial_p$$

Energy conservation: $\dot{H} = \{H, H\} = 0$ by skew-symmetry.

‘Phase space’ = Poisson manifold/variety

A **Poisson structure** on a space X is a bilinear operation on functions:

$$\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

satisfying the following axioms:

- skew-symmetry: $\{f,g\} = -\{g,f\}$
- derivation: $\{f,gh\} = \{f,g\}h + g\{f,h\}$
- Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Flows generated by functions, generalizing classical mechanics:

$H \in \mathcal{O}_X \qquad \Longrightarrow \qquad \xi_H := \{H,-\} \in \mathcal{T}_X$ **Hamiltonian vector field**

Conservation law: $\dot{H} = \xi_H(H) = \{H,H\} = 0$

Poisson geometry (1970s–) = study of Poisson manifolds, connections with classical/quantum mechanics, dynamics, integrable systems, foliations, Lie theory, representation theory, algebraic geometry, singularity theory, gauge theory, moduli spaces, noncommutative algebra, ...

Plan for the mini-course

Four-lecture introduction to $\mathbb{C}$-analytic/algebraic Poisson varieties:

I Basic Poisson geometry

II Foliations of Poisson varieties

III Global study: regular case

IV Global study: Fano case

Prerequisites:

- (complex) manifolds/varieties
- foliations

Optional but helpful:

- Lie algebras/groups
- symplectic geometry

Conventions

X complex analytic/algebraic variety (manifold, scheme...)

For simplicity: X smooth, but this can be dropped in many places

Standard notations for sheaves:

- $\mathcal{O}_X$ holomorphic functions
- $\mathcal{T}_X$ holomorphic vector fields $=$ derivations of $\mathcal{O}_X$
- $\Omega^\bullet_X$ holomorphic differential forms

Polyvectors

$$\mathcal{T}_X = \text{(holomorphic) vector fields} = \text{derivations of } \mathcal{O}_X$$

$$\wedge^p \mathcal{T}_X = p\text{-vector fields} = \text{skew multiderivations } \underbrace{\mathcal{O}_X \times \cdots \times \mathcal{O}_X}_{p \text{ times}} \to \mathcal{O}_X$$

Warning: if X singular, replace $\wedge^p \mathcal{T}_X$ everywhere with $(\Omega_X^p)^\vee$

Schouten bracket:

$$[-,-] : \wedge^p \mathcal{T}_X \times \wedge^q \mathcal{T}_X \to \wedge^{p+q-1} \mathcal{T}_X$$

extending Lie bracket on $\mathcal{T}_X$ = commutator of derivations, e.g. symmetric

$$\wedge^2 \mathcal{T}_X \times \wedge^2 \mathcal{T}_X \to \wedge^3 \mathcal{T}_X$$

skew + derivation $\iff \{f,g\} = \langle \pi, df \wedge dg \rangle$ where $\pi \in H^0(\wedge^2 \mathcal{T}_X)$

coordinates x^i $\quad \pi = \sum_{i<j} \pi^{ij} \partial_{x^i} \wedge \partial_{x^j} \quad \pi^{ij} = \{x^i, x^j\} \in \mathcal{O}_X$

$$\{f,g\} = \sum_{i<j} \pi^{ij} \cdot \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial x^j} \right)$$

Jacobi identity $\iff Jac(f,g,h) = 0 \ \forall f,g,h \iff [\pi,\pi] = 0 \in H^0(\wedge^3 \mathcal{T}_X)$

$$\underbrace{\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}}_{=:Jac(f,g,h)} = -\tfrac{1}{2} \langle [\pi,\pi], df \wedge dg \wedge dh \rangle$$

$$[\pi,\pi] = -2 \sum_{i<j<k} Jac(x^i,x^j,x^k) \partial_{x^i} \wedge \partial_{x^j} \wedge \partial_{x^k}$$

Poisson bivectors

$$\text{Poisson bracket } \{-,-\} \iff \begin{cases} \pi \in H^0(\wedge^2 \mathcal{T}_X) \\ 0 = [\pi,\pi] \in H^0(\wedge^3 \mathcal{T}_X) \end{cases}$$

Have **anchor map**

$$\begin{aligned} \pi^\sharp : \Omega^1_X &\to \mathcal{T}_X \\ \alpha &\mapsto \iota_\alpha \pi \end{aligned}$$

Hamiltonian vector field:

$$\xi_f = \{f,-\} = \langle \pi, df \wedge - \rangle = \iota_{df}\pi = \pi^\sharp(df)$$

Examples: constant brackets

$$X = \mathbb{C}^n \qquad \{x^i, x^j\} = \lambda^{ij} \in \mathbb{C} \qquad \pi = \sum_{i<j} \lambda_{ij} \partial_{x^i} \wedge \partial_{x^j}$$

$$\{x^i, \{x^j, x^k\}\} = \{x^i, \lambda^{jk}\} = 0 \implies \text{Jacobi identity}$$

More invariantly:

- $X = V$ vector space, $\dim V = n$
- $\pi \in \wedge^2 V \subset H^0(\wedge^2 \mathcal{T}_V)$ constant bivector $\implies [\pi, \pi] = 0$

Theorem

Any $\pi \in \wedge^2 V$ has even rank, say $\mathrm{rank}(\pi) = 2r$. Moreover, there exist linear coordinates q^i, p_i, $1 \le i \le r$ and z^j, $1 \le j \le n - 2r$ such that

$$\pi = \sum_{i=1}^{r} \partial_{q^i} \wedge \partial_{p_i},$$

$$\{q^i, p_j\} = \delta^i_j \qquad \{q^i, q^j\} = \{p_i, p_j\} = \{z^j, -\} = 0$$

Examples: linear brackets

Linear bracket on $X = \mathbb{C}^n$

$$\{x^i, x^j\} = \sum c_k^{ij} x^k$$

$$\mathfrak{g} := span(x^1, \ldots, x^n) \subset \mathcal{O}(X) \text{ a Lie algebra}$$

Lemma

$$\{\textit{Lie algebras}\} \cong \{\textit{vector spaces w/ linear Poisson structures}\}$$
$$\mathfrak{g} \leftrightarrow \mathfrak{g}^\vee = X$$

Example

$$\mathfrak{g} = \mathfrak{sl}(2;\mathbb{C}) \qquad \text{basis: } e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[e,f] = h \qquad [h,e] = 2e \qquad [h,f] = -2f$$

View (e, f, h) as coordinates on $X = \mathfrak{g}^\vee \cong \mathbb{C}^3$:

$$\pi = h\partial_e \wedge \partial_f + 2e\partial_h \wedge \partial_e - 2f\partial_h \wedge \partial_f$$

Definition

A bivector $\pi \in H^0(\wedge^2 \mathcal{T}_X)$ is **nondegenerate** if it defines a nondegenerate bilinear form on each cotangent space $T_p^* X$, $p \in X$.

$$\begin{aligned} \pi \text{ is nondegenerate} \quad &\iff \dim X = 2n \text{ and } \pi^n \text{ nonvanishing} \\ &\iff \pi^\sharp : \Omega^1_X \to \mathcal{T}_X \text{ an isomorphism} \\ &\iff \exists \pi^{-1} = \omega \in H^0(\Omega^2_X) \text{ nondegenerate} \end{aligned}$$

$$[\pi, \pi] = 0 \iff d\omega = 0$$

$$\{\text{nondegenerate Poisson structures } \pi\} \cong \{\text{holomorphic symplectic forms } \omega\}$$

Theorem (Darboux)

Near any point there exist coordinates q^i, p_i such that $\pi = \sum_{i=1}^n \partial_{q^i} \wedge \partial_{p_i}$

Example: smooth surfaces

Case $\dim_{\mathbb{C}} X = 2$:

- $\wedge^2 \mathcal{T}_X = \det \mathcal{T}_X =: K_X^{-1}$

$$\pi \in H^0(K_X^{-1})$$

locally $\pi = f(x^1, x^2)\partial_{x^1} \wedge \partial_{x^2}$

- $\wedge^3 \mathcal{T}_X = 0 \implies [\pi, \pi] = 0$
- zero locus: anticanonical divisor

$$Y = Zeros(\pi) \subset X$$

curve if nonempty

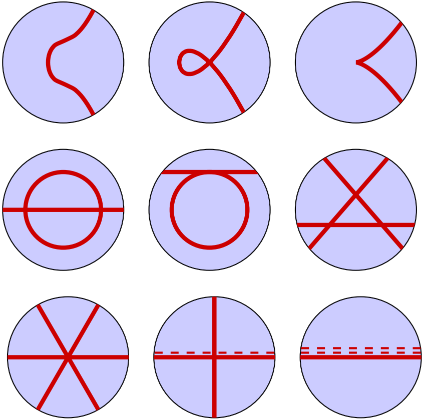
- Nondegenerate on $X \setminus Y$:

$$\pi \cong \partial_q \wedge \partial_p$$

- On smooth locus of Y:

$$\pi \cong u\partial_u \wedge \partial_v$$

$X = \mathbb{P}^2 \quad deg(K_X^{-1}) = 3 \quad Y =$ cubic



Morphisms

Definition

A map $\phi : (Y, \pi_Y) \to (X, \pi_X)$ is a **Poisson map** if

$$\phi^*\{f, g\}_X = \{\phi^* f, \phi^* g\}_Y$$

for all $f, g \in \mathcal{O}_X$. Equivalently: $\phi_* \pi_Y = \pi_X$.

Example (Maps to curves)

$\dim X = 1 \implies$ any $\phi : (Y, \pi_Y) \to (X, \pi_X = 0)$ is Poisson

Example (Open subvarieties)

Inclusion of any open set $(U, \pi|_U) \subset (X, \pi)$ is Poisson

Example (Closed subvarieties)

closed $Y \hookrightarrow (X, \pi)$ is Poisson $\iff$ $\{-,-\}_X$ descends to $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{I}_Y$
$\iff$ $\{\mathcal{I}_Y, \mathcal{O}_X\} \subset \mathcal{I}_Y$ $\iff$ π tangent to Y

More Poisson maps

Example (Products)

$(X, \pi_X), (Y, \pi_Y)$ Poisson $\rightsquigarrow (X \times Y, \pi_X + \pi_Y)$ Poisson, with Poisson maps

$$X \leftarrow X \times Y \rightarrow Y$$

Example (Quotients)

Suppose $G \circlearrowright (X, \pi)$ by Poisson isomorphisms such that X/G a variety.
Then $\exists!$ Poisson structure on X/G such that $X \to X/G$ is Poisson.

Local structure

Theorem (Weinstein splitting, J. Differ. Geom. 1983)

Suppose that $\pi \in \wedge^2 \mathcal{T}_X$ has rank $2r$ at a point $x \in X$. Then there are coordinates q^i, p_i and z_j centred at x such that

$$\pi = \sum_{i=1}^{r} \partial_{q^i} \wedge \partial_{p_i} + \underbrace{\sum_{1 \le j < k \le n-2r} g^{jk}(z) \partial_{z_j} \wedge \partial_{z_k}}_{\tilde{\pi}} \qquad g^{jk}|_x = 0$$

Moreover, germ of transverse Poisson structure $\tilde{\pi}$ is unique up to iso.

Corollary

Every point in (X, π) has a neighbourhood isomorphic to the product of a symplectic structure and a Poisson structure that vanishes at a point

Corollary (Darboux, Lie)

If $\mathrm{rank}(\pi)$ is constant near $x \in X$, then locally $\pi = \sum_{i=1}^{r} \partial_{q^i} \wedge \partial_{p_i}$

Existence of Weinstein splitting

Need to show: if $rank(\pi|_x) = 2r$, then

$$\pi = \sum_{i=1}^{r} \partial_{q^i} \wedge \partial_{p_i} + \sum g^{jk}(z)\partial_{z_j} \wedge \partial_{z_k} \qquad g^{jk}|_x = 0$$

Sketch of proof (induction on rank).

If $rank(\pi|_x) = 0$, there's nothing to prove, so assume $rank(\pi|_x) \geq 2$:

1. Choose $q, \tilde{p} \in \mathcal{O}_{X,x}$ with $\{q, \tilde{p}\}|_x \neq 0$, thus $\xi_q|_x \neq 0$
2. Straighten out ξ_q: get p with $\xi_q(p) = 1$, so $\{q, p\} = 1$
3. Jacobi $\implies [\xi_q, \xi_p] = \xi_{\{q,p\}} = 0$
4. $\therefore \exists$ coords $(q, p, z^3, \ldots, z^n)$ such $\xi_q = \partial_p$ and $\xi_p = -\partial_q$
5. This gives $\{q, p\} = 1$ and $\{q, z\} = \{p, z\} = 0$
6. Claim: $\{z^i, z^j\}$ indep. of p, q, indeed:

   $$\partial_p\{z^i, z^j\} = \xi_q\{z^i, z^j\} = \{q, \{z^i, z^j\}\} = \{\{q, z^i\}, z^j\} + \{z^i, \{q, z^j\}\} = 0 + 0$$
7. $\therefore \pi = \partial_q \wedge \partial_p + \tilde{\pi}(z)$ with $rank(\tilde{\pi})|_x = rank(\pi)|_x - 2$ $\square$