MMP - An overview

/$\mathbb{C}$

X = projective variety of dim n.

<u>Idea</u> decompose X into 3 blocks

① Fano ($-K_X$ is ample)

② CY ($K_X \equiv 0$)

③ canonically polarised (K_X is ample)

$\boxed{n=2}$ X = smooth surface

Let $E \simeq \mathbb{P}^1 \subset X$ s.t. $E^2 = -1$

<u>Castelnuovo</u> $\exists\ \varphi : X \to Y$ birational

s.t. $\operatorname{Exc} \varphi = E$.

Y is smooth surface.

If $\nexists$ such E then

① K_X is nef $K_X \cdot C \geq 0 \quad \forall C \geq 0$

(X minimal model)

OR

(2) $X \simeq \mathbb{P}^2$ or $\exists\, \eta : X \to C$ = curve

$\mathbb{P}^1$ (general fibre)

Mori's Program ($n \geq 3$)

X = smooth proj variety.

- Assume K_X is nef $\Rightarrow$ Stop X is minimal

Conjecture (OK for $n \leq 3$)

$\exists\, \eta : X \to Z$ s.t. $\exists\, m > 0$ H ample on Z

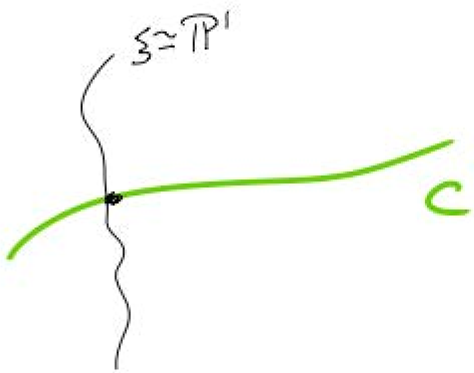
s.t. $mK_X = \eta^* H$

- If K_X is not nef. $\Rightarrow \exists\, C$ curve s.t.

$K_X \cdot C < 0$

Mori's Bend and Break $\forall x \in C$

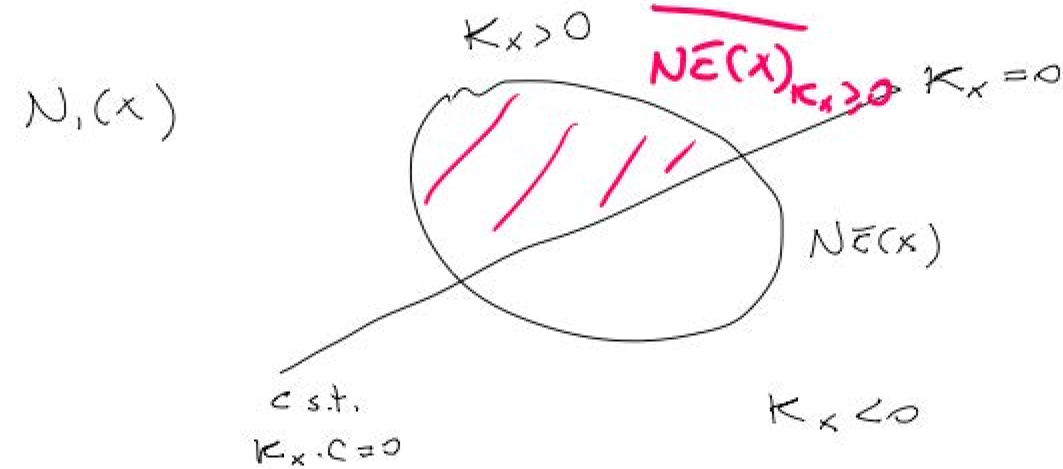
$\exists\, \xi \simeq \mathbb{P}^1 \ni x$ s.t. $K_X \cdot \xi < 0$



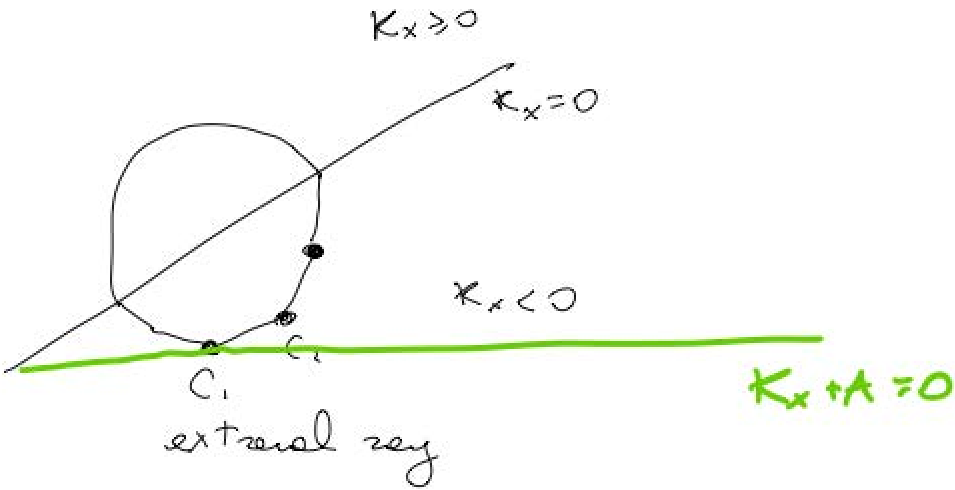
Cone theorem: $N_1(X) = \mathbb{R}$-vector space generated by the curves of $X / \equiv$

$C_1 \equiv C_2$ if $D \cdot C_1 = D\, C_2 \quad \forall D$

$$NE(X) = \left\{ \sum a_i [C_i] \mid a_i \geq 0 \quad C_i = \text{curve} \right\}$$
$$\subset N_1(X)$$



Theorem: (Mori) $\exists\ C_1, C_2, \ldots$ rational curves s.t. $\overline{NE(X)} = \overline{NE(X)}_{K_X \geq 0} + \sum_{i \geq 1} \mathbb{R}_{\geq 0} [C_i]$



Fix $\overline{C} = C_i$ for some i

<u>Corollary</u> $\exists$ A ample $\mathbb{Q}$-divisor s.t.

$(K_X + A) \cdot C \geq 0 \quad \forall C \in \overline{NE(X)}$

and $=$ holds iff $C \in \mathbb{R}_{\geq} [C]$

<u>Base pt free theorem</u> (Kawamata, Shokurov)

Set up as above, $K_X + A$ is semiample

$\exists$ $\boxed{\varphi : X \to Z}$ s.t. $m(K_X + A) = \varphi^*(H)$

$m > 0$ $\quad$ H ample on Z.

in particular, $\varphi(C) = pt$ iff $C \in \mathbb{R}_+ [\overline{C}]$

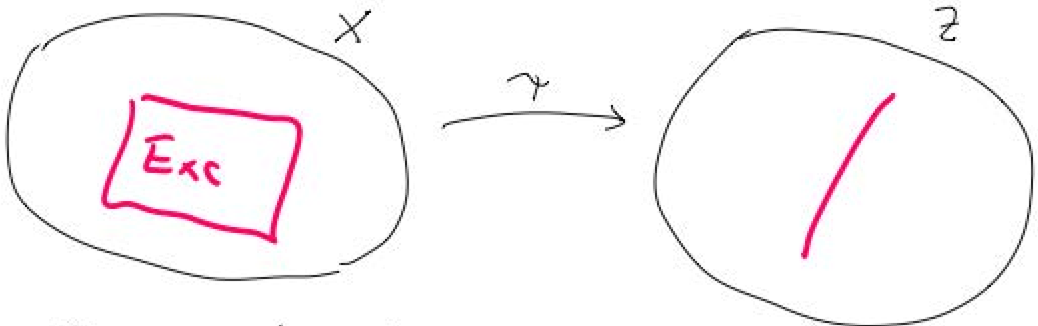
<u>1st case</u> $\dim Z < \dim X$

φ is called Mori fibre space.

the general fibre of φ is a Fano variety.

<u>2nd case</u> $\dim Z = \dim X$ i.e. φ is birational

case a) $\operatorname{Exc} \varphi$ is a divisor $\qquad \operatorname{Exc}\varphi = \bigcup_{C \in \mathbb{R}_+[C]} C$

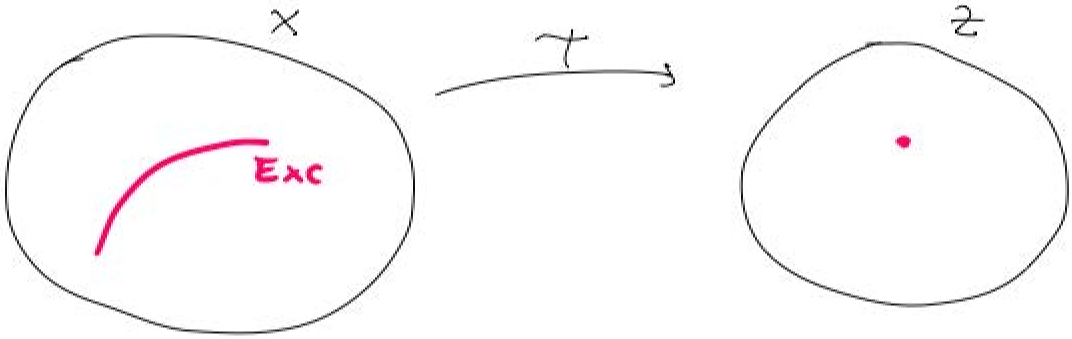


Z might have some singularities

(but mild) called <u>terminal singularities</u>

$\Rightarrow$ Replace X by Z $\boxed{\text{Start again}}$

<u>Case b)</u> $\operatorname{Exc}\varphi$ has $\operatorname{codim} \geq 2$



In this case

Z is <u>always</u> very singular $\quad mK_Z$ is not Cartier for all $m>0$

<u>but</u> $\exists$ <u>flip</u>

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\psi"'] & & X' \arrow[dl, "\psi'"] \\ & Z & \end{tikzcd}$$

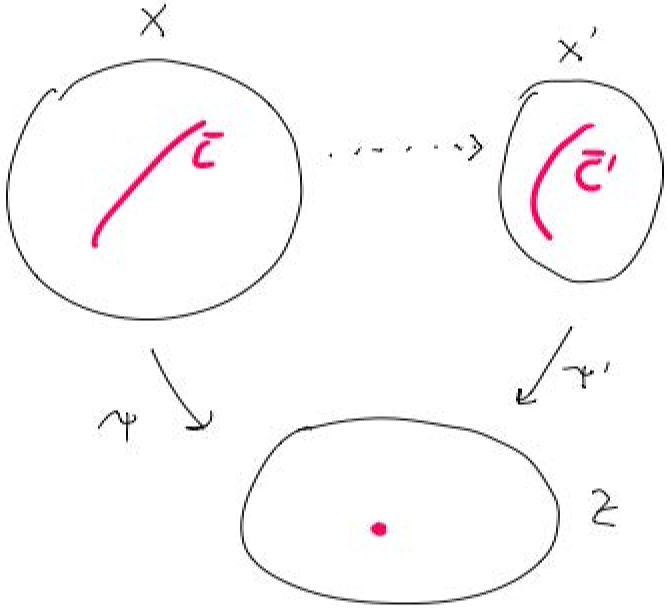
ψ' is birational

Exc ψ' has codim ≥ 2

and $\psi(C') = pt$ iff

$C' \in \mathbb{R}_{\geq 0}[\bar{C}']$ for some

C' s.t. $\boxed{K_{X'} \cdot C' > 0}$



X' is terminal

We can replace X by X'.

Start again

Why does this stop?

Open problem (Termination)

$n=3$ ok (Shokurov)

$\boxed{n \geq 3}$ $\exists$ special sequence of flips terminates if one of the following holds

(A) X = uniruled (X is covered by rational curves)

(we end up with a Mori fibre space)

(B) X = general type (K_X is big, i.e. $h^0(mK_X) \simeq c\, m^n$, $c>0$, $n = \dim X$)

(we end up with a minimal model)